doi: 10.17951/a.2019.73.1.33-39

ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LXXIII, NO. 1, 2019 SECTIO A 33–39

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Some inequalities for maximum modulus of rational functions

Abstract. In this paper, we establish some inequalities for rational functions with prescribed poles and restricted zeros in the sup-norm on the unit circle in the complex plane. Generalizations and refinements of rational function inequalities of Govil, Li, Mohapatra and Rodriguez are obtained.

1. Introduction. Let $\mathbb{P}_n$ denote the class of all complex algebraic polynomials $P(z)$ of degree n. For $a_j \in \mathbb{C}$ with $j = 1, 2, \dots, n$, let

$$W(z) := \prod_{j=1}^{n} (z - a_j)$$

and let

$$B(z) := \prod_{j=1}^{n} \Big(\frac{1 - \overline{a}_j z}{z - a_j}\Big), \quad \mathbb{R}_n := \mathbb{R}_n(a_1, a_2, \dots, a_n) = \Big\{ \frac{P(z)}{W(z)} : \ P \in \mathbb{P}_n \Big\}.$$

Then $\mathbb{R}_n$ is the set of rational functions with poles $a_1, a_2, \dots, a_n$ at most and with a finite limit at ∞. Note that $B(z) \in \mathbb{R}_n$ and $|B(z)| = 1$ for $|z| = 1$.

Definition 1.1. *(i)* For $P \in \mathbb{P}_n$, the conjugate transpose P^* of P is defined as $P^*(z) = z^n \overline{P\big(\frac{1}{\overline{z}}\big)}$.
(ii) For $r(z) = \frac{P(z)}{W(z)} \in \mathbb{R}_n$, the conjugate transpose r^* of r is defined as $r^*(z) = B(z)\overline{r\big(\frac{1}{\overline{z}}\big)}$.

2010 *Mathematics Subject Classification.* 30A10, 30C10, 30D15.
Key words and phrases. Rational function, polynomial, poles, zeros.

For $P \in \mathbb{P}_n$, we have

$$\max_{|z|=1} \left|P'(z)\right| \leq n \max_{|z|=1} |P(z)| \tag{1.1}$$

and

$$\max_{|z|=R\geq 1} |P(z)| \leq R^n \max_{|z|=1} |P(z)|. \tag{1.2}$$

The inequality (1.1) is the famous Bernstein's inequality (for reference see [3]) and (1.2) is an immediate consequence of the maximum modulus principle. Equality holds in (1.1) and (1.2) for $P(z) = \lambda z^n$, $\lambda \neq 0$. Noting that these extremal polynomials have all zeros at the origin, it is natural to seek improvements under appropriate conditions on the zeros of $P(z)$.

For $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, we have

$$\max_{|z|=1} \left|P'(z)\right| \leq \frac{n}{2} \max_{|z|=1} |P(z)| \tag{1.3}$$

and

$$\max_{|z|=R\geq 1} |P(z)| \leq \left(\frac{R^n + 1}{2}\right) \max_{|z|=1} |P(z)|. \tag{1.4}$$

Equality holds in (1.3) and (1.4) for $P(z) = \alpha z^n + \beta$, $|\alpha| = |\beta|$.

As is well known, the inequality (1.3) was conjectured by Erdös and proved by Lax [5] and the inequality (1.4) is due to Ankeny and Rivlin [1]. Further, Aziz and Dawood [2] sharpened the inequality (1.4) and proved that if $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then

$$\max_{|z|=R\geq 1} |P(z)| \leq \frac{R^n + 1}{2} \max_{|z|=1} |P(z)| - \frac{R^n - 1}{2} \min_{|z|=1} |P(z)|. \tag{1.5}$$

The estimate is sharp and equality holds in (1.5) for $P(z) = \alpha z^n + \beta$, $|\alpha| = |\beta|$.

In the past few years, several papers pertaining to Bernstein-type inequalities for rational functions have appeared in the study of rational approximation problems. In fact in 1995, Li, Mohapatra and Rodriguez [7] proved some inequalities similar to (1.1) and (1.3) for rational functions with poles outside the unit circle. They extended (1.1) to rational functions by proving that if $r \in \mathbb{R}_n$, then for $|z| = 1$,

$$|r'(z)| \leq |B'(z)| \max_{|z|=1} |r(z)|. \tag{1.6}$$

As an extension of (1.3) to rational functions, they also proved that if $r \in \mathbb{R}_n$ and all the zeros of $r(z)$ lie in $|z| \geq 1$, then for $|z| = 1$,

$$|r'(z)| \leq \frac{|B'(z)|}{2} \max_{|z|=1} |r(z)|. \tag{1.7}$$

Recently, Govil and Mohapatra [4] obtained inequalities analogous to (1.2) and (1.4) for the class of rational functions with poles outside the unit circle. In fact they proved that if $r \in \mathbb{R}_n$, then

$$|r(z)| \leq |B(z)| \max_{|z|=1} |r(z)|, \ |z| \geq 1 \tag{1.8}$$

and if all the zeros of $r(z)$ lie in $|z| \geq 1$, then

$$|r(z)| \leq \left(\frac{|B(z)|+1}{2}\right) \max_{|z|=1} |r(z)|, \ |z| \geq 1. \tag{1.9}$$

It may be noted that inequalities (1.2) and (1.4) can be deduced from inequalities (1.8) and (1.9) respectively by multiplying the two sides of (1.8) and (1.9) by $\prod_{\nu=1}^{n} a_\nu$ and then let each a_ν go to infinity.

Our main aim is to obtain an inequality analogous to inequality (1.5) for rational functions with poles outside the unit circle as considered by Li, Mohapatra and Rodriguez [7], but our method of proof is different from the method of Li, Mohapatra and Rodriguez.

2. Main results. In this section we state our main results. Their proofs are given in the next section. From now on, we shall always assume that all the poles $a_1, a_2, \ldots, a_n$ lie in $|z| > 1$. Our first result that is presented below provides a generalization of (1.7).

Theorem 2.1. *If $r \in \mathbb{R}_n$ and all the n zeros of $r(z)$ lie in $|z| \geq 1$, then for every β with $|\beta| \leq 1$ and $|z| = 1$,*

$$\left|B(z)r'(z) + \frac{\beta}{2}B'(z)r(z)\right| \leq \frac{|B'(z)|}{2}\left\{\left|1+\frac{\beta}{2}\right| + \left|\frac{\beta}{2}\right|\right\} \max_{|z|=1} |r(z)|. \tag{2.1}$$

Remark 2.1. For $\beta = 0$, (2.1) reduces to (1.7).

Our next result that provides an inequality analogous to (1.5) for rational functions is given by

Theorem 2.2. *If $r \in \mathbb{R}_n$ and all the zeros of $r(z)$ lie in $|z| \geq 1$, then for $|z| \geq 1$,*

$$|r(z)| \leq \left(\frac{|B(z)|+1}{2}\right) \max_{|z|=1} |r(z)| - \left(\frac{|B(z)|-1}{2}\right) \min_{|z|=1} |r(z)|. \tag{2.2}$$

Equality holds in (2.2) for $r(z) = \alpha B(z) + \beta, \ |\alpha| = |\beta|$.

3. Lemmas. For the proofs of our theorems we need the following lemmas.

Lemma 3.1. *If $r \in \mathbb{R}_n$ has n zeros which all lie in $|z| \leq 1$, then*

$$|r'(z)| \geq \frac{1}{2}|B'(z)||r(z)| \ for \ |z| = 1. \tag{3.1}$$

Equality holds in (3.1) for $r(z) = \mu B(z) + \zeta$ with $|\mu| = |\zeta| = 1$.

The above lemma is due to Li, Mohapatra and Rodriguez [7].

Lemma 3.2. *Let A and B be any two complex numbers. Then*
(i) if $|A| \geq |B|$ and $B \neq 0$, then $A \neq \delta B$ for all complex numbers δ satisfying $|\delta| < 1$.
(ii) Conversely, if $A \neq \delta B$ for all complex numbers δ satisfying $|\delta| < 1$, then $|A| \geq |B|$.

The above lemma is due to Li [6].

Lemma 3.3. *If $r \in \mathbb{R}_n$ and $|z| = 1$, then for every β with $|\beta| \leq 1$,*

$$\left|B(z)r'(z) + \frac{\beta}{2}B'(z)r(z)\right| + \left|B(z)(r^*(z))' + \frac{\beta}{2}B'(z)r^*(z)\right| \leq |B'(z)|\left\{\left|1 + \frac{\beta}{2}\right| + \left|\frac{\beta}{2}\right|\right\}\max_{|z|=1}|r(z)|.$$

Proof of Lemma 3.3. Let $M := \max_{|z|=1}|r(z)|$. Therefore, for every λ with $|\lambda| > 1$, $|r(z)| < |\lambda M B(z)|$ for $|z| = 1$. By Rouché's theorem, all the zeros of $G(z) = r(z) + \lambda M B(z)$ lie in $|z| < 1$. If $H(z) = B(z)\overline{G(\frac{1}{\bar{z}})}$, then $|H(z)| = |G(z)|$ for $|z| = 1$ and hence for any γ with $|\gamma| < 1$, the rational function $\gamma H(z) + G(z)$ has all zeros in $|z| < 1$. By applying Lemma 3.1 to $\gamma H(z) + G(z)$, we have

$$2\big|B(z)\big(\gamma H'(z) + G'(z)\big)\big| \geq |B'(z)|\big|\gamma H(z) + G(z)\big|, \tag{3.2}$$

for $|z| = 1$. Since $B'(z) \neq 0$, so the right hand side of (3.2) is non zero. Thus, by using (i) of Lemma 3.2, we have for all $\beta \in \mathbb{C}$ with $|\beta| < 1$,

$$2B(z)\big(\gamma H'(z) + G'(z)\big) \neq -\beta B'(z)\big(\gamma H(z) + G(z)\big),$$

for $|z| = 1$. Equivalently, for $|z| = 1$,

$$-\gamma\big(2B(z)H'(z) + \beta B'(z)H(z)\big) \neq \big(2B(z)G'(z) + \beta B'(z)G(z)\big), \tag{3.3}$$

for $|\gamma| < 1$ and $|\beta| < 1$.

Using (ii) of Lemma 3.2 in (3.3), we have

$$\big|2B(z)G'(z) + \beta B'(z)G(z)\big| \geq \big|2B(z)H'(z) + \beta B'(z)H(z)\big|, \tag{3.4}$$

for $|z| = 1$ and $|\beta| < 1$. Now by putting $G(z) = r(z) + \lambda M B(z)$ and $H(z) = r^*(z) + \overline{\lambda}M$ in (3.4), we get for $|z| = 1$ and $|\beta| < 1$,

$$\begin{aligned} &\big|2B(z)(r^*(z))' + \beta B'(z)r^*(z) + \overline{\lambda}\beta M B'(z)\big| \\ &\qquad \leq \big|2B(z)r'(z) + \beta B'(z)r(z) + \lambda B(z)B'(z)\big(2 + \beta\big)M\big|. \end{aligned} \tag{3.5}$$

By choosing a suitable argument of λ on the right hand side of (3.5), we get for $|z| = 1$ and $|\beta| < 1$,

$$\begin{aligned} &\big|2B(z)(r^*(z))' + \beta B'(z)r^*(z)\big| - |\lambda|\big|\beta B'(z)\big|M \\ &\qquad \leq |\lambda|\big|B(z)B'(z)\big(2 + \beta\big)\big|M - \big|2B(z)r'(z) + \beta B'(z)r(z)\big|. \end{aligned} \tag{3.6}$$

Note that $|B(z)| = 1$ for $|z| = 1$. Making $|\lambda| \to 1$ and using continuity for $|\beta| = 1$ in (3.6), we get the desired result. □

The following two lemmas are due to Govil and Mohapatra [4].

Lemma 3.4. *If $r \in \mathbb{R}_n$ and all the zeros of $r(z)$ lie in $|z| \geq 1$, then for $|z| \geq 1$,*

$$|r(z)| \leq |r^*(z)|.$$

Lemma 3.5. *If $r \in \mathbb{R}_n$, then for $|z| \geq 1$,*

$$|r(z)| + |r^*(z)| \leq (|B(z)| + 1) \max_{|z|=1} |r(z)|.$$

4. Proofs of Theorems.

Proof of Theorem 2.1. Since all the zeros of $r(z)$ lie in $|z| \geq 1$, therefore all the zeros of $r^*(z)$ lie in $|z| \leq 1$. First assume that no zeros of $r^*(z)$ lie on the unit circle $|z| = 1$ and therefore, that all the zeros of $r^*(z)$ are in $|z| < 1$. By Rouché's theorem, the rational function $\lambda r(z) + r^*(z)$ has all its zeros in $|z| < 1$ for $|\lambda| < 1$ and has no poles in $|z| \leq 1$. On applying Lemma 3.1 to $\lambda r(z) + r^*(z)$, we get on $|z| = 1$,

$$2|B(z)|\big|\lambda r'(z) + (r^*(z))'\big| \geq |B'(z)|\big|\lambda r(z) + r^*(z)\big|. \tag{4.1}$$

Note that $B'(z) \neq 0$ (e.g. see formula (14) in [7]). So, the right hand side of (4.1) is non zero. Thus, by using (i) of Lemma 3.2, we have for all $\beta \in \mathbb{C}$ with $|\beta| < 1$,

$$2B(z)\big(\lambda r'(z) + (r^*(z))'\big) \neq -\beta B'(z)\big(\lambda r(z) + r^*(z)\big)$$

for $|z| = 1$. Equivalently

$$\lambda\big(2B(z)r'(z) + \beta B'(z)r(z)\big) \neq -\big(2B(z)(r^*(z))' + \beta B'(z)r^*(z)\big) \tag{4.2}$$

for $|z| = 1$, $|\lambda| < 1$ and $|\beta| < 1$. Now using (ii) of Lemma 3.2 in (4.2), we have

$$\big|2B(z)r'(z) + \beta B'(z)r(z)\big| \leq \big|2B(z)(r^*(z))' + \beta B'(z)r^*(z)\big| \tag{4.3}$$

for $|z| = 1$ and $|\beta| < 1$. The above inequality (4.3) in conjunction with Lemma 3.3 proves Theorem 2.1 when $r^*(z)$ has no zero on $|z| = 1$ and $|\beta| < 1$. Now using the continuity in the zeros and β, we can obtain the inequality (2.1) of Theorem 2.1, when some zeros of $r^*(z)$ lie on the unit circle $|z| = 1$ and $|\beta| \leq 1$. □

Proof of Theorem 2.2. Let $m = m(r, 1) = \min_{|z|=1} |r(z)|$. If $r(z)$ has a zero on $|z| = 1$, then $m = 0$ and Theorem 2.2 is reduced to inequality (1.9), therefore, we assume that $r(z)$ has all its zeros in $|z| > 1$, so that $m > 0$. We have $|\lambda m| < |r(z)|$ on $|z| = 1$ for any λ with $|\lambda| < 1$. By Rouché's theorem, the rational function $G(z) = r(z) - \lambda m$ has no zero in $|z| < 1$. Therefore, the rational function $H(z) = B(z)\overline{G(\frac{1}{\bar{z}})} = r^*(z) - \overline{\lambda} m B(z)$ will have all its zeros in $|z| < 1$. Also $|H(z)| = |G(z)|$ for $|z| = 1$. On applying Lemma 3.4, we get for every $|z| \geq 1$,

$$|G(z)| \leq |H(z)|.$$

Equivalently

$$|r(z) - \lambda m| \leq |r^*(z) - \overline{\lambda} m B(z)|$$

for $|z| \geq 1$, which implies for every λ with $|\lambda| < 1$,

$$|r(z)| - |\lambda| m(r,1) \leq |r^*(z) - \overline{\lambda} m(r,1) B(z)| \tag{4.4}$$

for $|z| \geq 1$. Since $r^*(z) = B(z)\overline{r(\frac{1}{\overline{z}})}$ has all its zeros in $|z| < 1$ and

$$|m(r,1)B(z)| \leq |r(z)| = |r^*(z)|$$

for $|z| = 1$, so that

$$\frac{m(r,1)B(z)}{r^*(z)}$$

is analytic for $|z| \geq 1$. Hence by maximum modulus principle for unbounded domains, we have

$$|m(r,1)B(z)| \leq |r^*(z)|$$

for $|z| \geq 1$, we can choose the argument of λ so that the right hand side of (4.4) is

$$|r^*(z)| - |\lambda| m(r,1) |B(z)|. \tag{4.5}$$

Combining (4.4) and (4.5), we get

$$|r(z)| - |\lambda| m(r,1) \leq |r^*(z)| - |\lambda| m(r,1) |B(z)|,$$

which implies by letting $|\lambda| \to 1$,

$$|r(z)| \leq |r^*(z)| - (|B(z)| - 1) m(r,1)$$

for $|z| \geq 1$. The above inequality in conjunction with Lemma 3.5 gives for $|z| \geq 1$,

$$2|r(z)| \leq (|B(z)| + 1) \max_{|z|=1} |r(z)| - (|B(z)| - 1) m(r,1),$$

which is equivalent to (2.2) and this completes the proof of Theorem 2.2. □

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Received November 25, 2017