doi: 10.17951/a.2021.75.2.93-107

ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LXXV, NO. 2, 2021 SECTIO A 93–107

MACIEJ PAROL and DARIUSZ PARTYKA

Contribution to the Hadamard multiplication theorem

Abstract. In this article we define a binary linear operator T for holomorphic functions in given open sets A and B in the complex plane under certain additional assumptions. It coincides with the classical Hadamard product of holomorphic functions in the case where A and B are the unit disk. We show that the operator T exists provided A and B are simply connected domains containing the origin. Moreover, T is determined explicitly by means of an integral form. To this aim we prove an alternative representation of the star product $A * B$ of any sets $A, B \subset \mathbb{C}$ containing the origin. We also touch the problem of holomorphic extensibility of Hadamard product.

1. Introduction. Let $\mathrm{Hol}(\Omega)$ denote the class of all holomorphic functions in an open set Ω. Here and later on, we assume that all topological notions refer to the extended complex plane $\mathrm{E}(\hat{\mathbb{C}}) := (\hat{\mathbb{C}}, \rho)$, where ρ is the chordal metric. For any $a \in \mathbb{C}$ and $r > 0$ we set $\mathbb{D}(a, r) := \{z \in \mathbb{C} : |z - a| < r\}$ and $\overline{\mathbb{D}}(a, r) := \{z \in \mathbb{C} : |z - a| \le r\}$. In particular $\mathbb{D} := \mathbb{D}(0, 1)$ is the *unit disk* and $\overline{\mathbb{D}} := \overline{\mathbb{D}}(0, 1)$ is the *closed unit disk*. The *Hadamard product of functions f and g holomorphic in a certain neighbourhood of the origin* is given by the formula

$$\mathbb{D}(0, R_{f,g}) \ni z \mapsto f * g(z) := \sum_{n=0}^{+\infty} \frac{f^{(n)}(0) g^{(n)}(0)}{(n!)^2} z^n, \tag{1.1}$$

2010 *Mathematics Subject Classification.* 47A05, 30B10, 30B40.

Key words and phrases. Hadamard product, holomorphic extension, star product, Hadamard multiplication theorem.

where

$$R_{f,g} := \left(\limsup_{n\to+\infty} \sqrt[n]{\frac{|f^{(n)}(0)|}{n!} \cdot \frac{|g^{(n)}(0)|}{n!}} \right)^{-1}; \tag{1.2}$$

see e.g. [2], [3]. For example, using the formulas (1.1) and (1.2) for the functions $\mathbb{C}\setminus\{1\} \ni z \mapsto f(z) := 1/(1-z)$ and $g := f$ we compute $f*g = f|_{\mathbb{D}}$. On the other hand the Hadamard product $f*g$ has a holomorphic extension to the function f. This means that in certain cases the Hadamard product $f*g$ can be extended holomorphically outside the disk $\mathbb{D}(0, R_{f,g})$. Thus for given domains $A \subset \mathbb{C}$ and $B \subset \mathbb{C}$ containing the origin we seek domains Ω such that for all $f \in \mathrm{Hol}(A)$ and $g \in \mathrm{Hol}(B)$ the Hadamard product $f*g$ has a holomorphic extension to a domain containing Ω. To specify this issue write

$$\rho(D) := \sup(\{r \geq 0 : \mathbb{D}(0,r) \subset D\}) \tag{1.3}$$

for a set $D \subset \mathbb{C}$ containing 0. Notice that for all domains $A \subset \mathbb{C}$ and $B \subset \mathbb{C}$ containing the origin,

$$\rho(A)\rho(B) \leq R_{f,g}, \quad f \in \mathrm{Hol}(A),\ g \in \mathrm{Hol}(B). \tag{1.4}$$

Given domains $A \subset \mathbb{C}$ and $B \subset \mathbb{C}$ containing the origin we consider the class $\mathcal{H}_1(A,B)$ of all domains $\Omega \subset \mathbb{C}$ such that $D := \mathbb{D}(0, \rho(A)\rho(A)) \subset \Omega$ and for all $f \in \mathrm{Hol}(A)$ and $g \in \mathrm{Hol}(B)$ there exists $h \in \mathrm{Hol}(\Omega)$ which coincides with $f*g$ in D, i.e.,

$$h(z) = f*g(z),\ z \in \mathbb{D}(0, \rho(A)\rho(B)). \tag{1.5}$$

In other words the function $(f*g)|_D$ has a holomorphic extension h to each $\Omega \in \mathcal{H}_1(A,B)$. Thus for each $\Omega \in \mathcal{H}_1(A,B)$ and a domain Ω', if $\mathbb{D}(0,\rho(A)\rho(B)) \subset \Omega' \subset \Omega$, then $\Omega' \in \mathcal{H}_1(A,B)$. Therefore we can ask about the maximal $\Omega \in \mathcal{H}_1(A,B)$ in the sense of inclusion. It is easy to find an upper bound of the class $\mathcal{H}_1(A,B)$. Fix $u \in \mathbb{C}\setminus A$ and $v \in \mathbb{C}\setminus B$. Setting

$$\mathbb{C}\setminus\{u\} \ni z \mapsto f(z) := \frac{1}{1-\frac{z}{u}} \ \text{ and } \ \mathbb{C}\setminus\{v\} \ni z \mapsto g(z) := \frac{1}{1-\frac{z}{v}}$$

we see that $f \in \mathrm{Hol}(A)$ and $g \in \mathrm{Hol}(B)$. Moreover,

$$f(z) = \sum_{n=0}^{+\infty} \frac{1}{u^n} z^n,\ z \in \mathbb{D}(0,|u|) \ \text{ and } \ g(z) = \sum_{n=0}^{+\infty} \frac{1}{v^n} z^n,\ z \in \mathbb{D}(0,|v|),$$

from which

$$f*g(z) = \sum_{n=0}^{+\infty} \frac{1}{(uv)^n} z^n = \frac{1}{1-\frac{z}{uv}},\ z \in \mathbb{D}(0,|uv|).$$

Thus $uv \notin \Omega$ for $\Omega \in \mathcal{H}_1(A,B)$, and consequently,

$$(\hat{\mathbb{C}}\setminus A)\cdot(\hat{\mathbb{C}}\setminus B) \subset \hat{\mathbb{C}}\setminus\Omega,\ \Omega \in \mathcal{H}_1(A,B), \tag{1.6}$$

where

$$X \cdot Y := \big\{xy : x \in X \text{ and } y \in Y \text{ and } \big(x \neq 0 \neq y \text{ or } x \neq \infty \neq y\big)\big\}$$

for all nonempty sets $X, Y \subset \hat{\mathbb{C}}$. We put here $c \cdot \infty := \infty$ and $\infty \cdot c := \infty$ for $c \in \hat{\mathbb{C}} \setminus \{0\}$. Additionally we set $\emptyset \cdot X := \emptyset$ and $X \cdot \emptyset := \emptyset$. Setting for all sets $X, Y \subset \hat{\mathbb{C}}$,

$$X^{\complement} := \hat{\mathbb{C}} \setminus X \quad \text{and} \quad X * Y := (X^{\complement} \cdot Y^{\complement})^{\complement}$$

we can rewrite the condition (1.6) as follows

$$\Omega \subset A * B := (A^{\complement} \cdot B^{\complement})^{\complement}, \ \Omega \in \mathcal{H}_1(A, B). \tag{1.7}$$

In this manner the binary operation $*$ was defined on the class $2^{\hat{\mathbb{C}}}$. The set $A * B$ is usually called the *star product of the sets* $A, B \subset \hat{\mathbb{C}}$. For example for all $r_1, r_2 > 0$ we have

$$\mathbb{D}(0, r_1) * \mathbb{D}(0, r_2) = \mathbb{D}(0, r_1 r_2). \tag{1.8}$$

Since $\Omega \in \mathcal{H}_1(A, B)$ is a connected set we deduce from (1.7) that

$$\Omega \subset \mathrm{CC}_0(A * B), \ \Omega \in \mathcal{H}_1(A, B), \tag{1.9}$$

where $\mathrm{CC}_z(X)$ denotes the connected component of a set $X \subset \hat{\mathbb{C}}$ containing z, i.e.

$$\mathrm{CC}_z(X) := \bigcup_{V \in \mathcal{F}_z} V,$$

where

$$\mathcal{F}_z := \big\{V \in 2^{\hat{\mathbb{C}}} : z \in V \subset X \text{ and } V \text{ is connected}\big\}.$$

The problem of holomorphic extending of $(f * g)|_D$ to domains depending on A and B only is fairly old; cf. [3]. Recently, the problem was studied by Grosse-Erdmann in [1], Lorson in [6] and [7] as well as by Müller and Pohlen in [9]. The natural question is whether the equality can appear in the inclusion (1.9). The final solution to this problem was given in 1992 by Müller. He proved in [8] that for all domains A and B containing 0, $\mathrm{CC}_0(A*B) \in \mathcal{H}_1(A, B)$, i.e., $\mathrm{CC}_0(A*B)$ is the maximal domain in $\mathcal{H}_1(A, B)$ in the sense of inclusion. He called this result *Hadamard Multiplication Theorem*. If domains A and B are simply connected, then the extension h of $(f * g)|_D$ can be characterized in a more constructive way given in Corollary 3.2. This is a conclusion from Theorem 3.1, which deals with the following concept of generalization of the Hadamard product operator $*$. Given non-empty open sets $A, B \subset \mathbb{C}$ we denote by $\mathcal{H}_2(A, B)$ the class of all open sets $\Omega \subset \mathbb{C}$ such that there exists an operator $T : \mathrm{Hol}(A) \times \mathrm{Hol}(B) \to \mathrm{Hol}(\Omega)$ satisfying the following conditions:

(1.10) $T(f, g) = f * g$ for all polynomials f and g;

(1.11) For all sequences $\mathbb{N} \ni n \mapsto f_n \in \mathrm{Hol}(A)$ and $\mathbb{N} \ni n \mapsto g_n \in \mathrm{Hol}(B)$ and all $f \in \mathrm{Hol}(A)$ and $g \in \mathrm{Hol}(B)$, if $f_n \xrightarrow{\mathrm{ucc}} f$ in A as $n \to +\infty$ and $g_n \xrightarrow{\mathrm{ucc}} g$ in B as $n \to +\infty$, then $T(f_n, g_n) \xrightarrow{\mathrm{ucc}} T(f, g)$ in Ω as $n \to +\infty$.

Here and later on the symbol $\xrightarrow{\mathrm{ucc}}$ stands for the uniform convergence on compact sets. For example

$$\Omega := \mathbb{D}(0, r_1 r_2) \in \mathcal{H}_2(\mathbb{D}(0, r_1), \mathbb{D}(0, r_2)),\ r_1, r_2 > 0. \tag{1.12}$$

To prove this property notice that for given $r_1, r_2 > 0$ we deduce from (1.2) that $R_{f,g} \geq r_1 r_2$ for all $f \in \mathrm{Hol}(\mathbb{D}(0, r_1))$ and $g \in \mathrm{Hol}(\mathbb{D}(0, r_2))$. Then $\Omega \subset \mathbb{D}(0, R_{f,g})$, and setting

$$\mathrm{Hol}(\mathbb{D}(0, r_1)) \times \mathrm{Hol}(\mathbb{D}(0, r_2)) \ni (f, g) \mapsto T(f, g) := f * g$$

we conclude from (1.1) that $T(f, g) \in \mathrm{Hol}(\Omega)$ for all $f \in \mathrm{Hol}(\mathbb{D}(0, r_1))$ and $g \in \mathrm{Hol}(\mathbb{D}(0, r_2))$ and the condition (1.10) holds. The second condition can be derived from the representation of the Hadamard product by the integral formula (1.13), called the *Parseval integral*; cf. [3, p. 84]. To be more specific the following theorem holds.

Theorem A. *For all $f, g \in \mathrm{Hol}(\mathbb{D})$ and $z \in \mathbb{D}$ the equality*

$$f * g(z) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}^+(0,r)} f(u) g\left(\frac{z}{u}\right) \frac{\mathrm{d}u}{u} \tag{1.13}$$

holds for $r \in (|z|; 1)$, where $[0; 2\pi] \ni t \mapsto \mathbb{T}^+(a, r)(t) := a + r\mathrm{e}^{\mathrm{i}t}$ for any $a \in \mathbb{C}$ and $r > 0$.

We can adopt the integral (1.13) to our case by scaling the unit disk $\mathbb{D}$ to the disks $\mathbb{D}(0, r_1)$ and $\mathbb{D}(0, r_2)$, respectively. Then passing to the limit under the integral leads to the condition (1.11), and consequently the property (1.12) is true. Thus in general case of non-empty domains $A, B \subset \mathbb{C}$ the operator T can be naturally interpreted as a generalization of the Hadamard product operator $*$. We show that such an operator exists for any simply connected domains $A, B \subset \mathbb{C}$ containing the origin and $T(f, g) \in \mathrm{Hol}(A * B)$ for all $f \in \mathrm{Hol}(A)$ and $g \in \mathrm{Hol}(B)$. Moreover, the operator can be expressed explicitly with an integral formula similar to (1.13); see also [1, Theorem 3.4]. This is stated in Theorem 3.1, which is our main result. The proof of Theorem 3.1 appeals to several auxiliary facts proved in the second section. Theorem 2.1 is of special interest here, because it provides an alternative characterization of the product $A * B$ for any sets $A, B \subset \hat{\mathbb{C}}$.

It is worth noting that the considerations presented in the remaining part of our paper deal with the very simple case of the operator T satisfying the condition (1.10). They are also applicable to the situation where the Hadamard product in the condition (1.10) is replaced by a more general binary linear form. However, this subject will be developed elsewhere.

The authors would like to thank the reviewer for his helpful comments.

2. Auxiliary results. In this section we gather several facts required in the proof of Theorem 3.1, which is our main result.

Theorem 2.1. *For all sets $A, B \subset \mathbb{C}$ containing the origin the equality*

$$A * B = \left\{ z \in \mathbb{C} \setminus \{0\} : \hat{\mathbb{C}} = A \cup \frac{z}{B} \right\} \cup \{0\} \tag{2.1}$$

holds.

Proof. Fix sets $A, B \subset \mathbb{C}$ containing 0 and fix $z \in \hat{\mathbb{C}}$. Suppose that

$$z \in \left\{ \zeta \in \mathbb{C} \setminus \{0\} : \hat{\mathbb{C}} = A \cup \frac{\zeta}{B} \right\}.$$

Then $z \in \mathbb{C} \setminus \{0\}$ and

$$A \cup \frac{z}{B} = \hat{\mathbb{C}},$$

which is equivalent to

$$A^{\complement} \cap \frac{z}{B^{\complement}} = \emptyset.$$

Then for all $a \in A^{\complement}$ and $b \in B^{\complement}$, $a \neq z/b$, and so $z \neq ab$. Therefore $z \notin A^{\complement} \cdot B^{\complement}$, which gives $z \in A * B$. This proves the following inclusion

$$\left\{ \zeta \in \mathbb{C} \setminus \{0\} : \hat{\mathbb{C}} = A \cup \frac{\zeta}{B} \right\} \subset A * B \setminus \{0\}. \tag{2.2}$$

To show the inverse inclusion to (2.2) suppose that

$$z \notin \left\{ \zeta \in \mathbb{C} \setminus \{0\} : \hat{\mathbb{C}} = A \cup \frac{\zeta}{B} \right\}.$$

If $z \in \{0, \infty\}$, then $z \notin A * B \setminus \{0\}$, because $A \cup B \subset \mathbb{C}$. Otherwise, $z \in \mathbb{C} \setminus \{0\}$ and

$$A \cup \frac{z}{B} \neq \hat{\mathbb{C}},$$

which gives

$$A^{\complement} \cap \frac{z}{B^{\complement}} \neq \emptyset.$$

Hence there exists $w \in \mathbb{C} \setminus \{0\}$ such that $w \in A^{\complement} \cap z/B^{\complement}$. This means that $w \in A^{\complement}$ and $z/w \in B^{\complement}$. Therefore $z = w \cdot (z/w) \in A^{\complement} \cdot B^{\complement}$, and so $z \notin A * B \setminus \{0\}$. Using now the law of contraposition we obtain the inclusion inverse to (2.2). Both the inclusions yield the equality (2.1), because $0 \in A * B$. $\square$

Example 2.2. Let $A := B := \{z \in \mathbb{C} : \mathrm{Re}(z) < 1\}$. Then

$$\frac{z}{B} = \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}\Big(\frac{z}{2}, \frac{|z|}{2}\Big).$$

Then by Theorem 2.1 we see that for every $z \in \mathbb{C}$, $z \in A * B$ if and only if

$$\mathrm{Re}\left(\frac{z}{2} + \frac{|z|}{2}\right) < 1,$$

and this inequality is equivalent to
$$\mathrm{Re}(z) < 1 - \frac{1}{4}(\mathrm{Im}(z))^2.$$
Therefore
$$A * B = \Big\{ z \in \mathbb{C} : \mathrm{Re}(z) < 1 - \frac{1}{4}(\mathrm{Im}(z))^2 \Big\}.$$

Now we quote a classical statement formulated by Zygmunt Janiszewski; see [4, p. 506, Theorem 5].

Theorem B. *Let A and B be two open sets or two closed sets. If A, B are connected and the set $A \cap B$ is disconnected, then the set $A \cup B$ separates a certain pair of points $p, q \in \hat{\mathbb{C}}$, i.e. p and q belong to different components of the set $(A \cup B)^{\complement}$.*

Using Theorem B, we can prove the following lemma.

Lemma 2.3. *Let Ω_1 and Ω_2 be simply connected domains such that $0 \in \Omega_1^{\complement}$, $\infty \in \Omega_2^{\complement}$ and $\Omega_1 \cup \Omega_2 = \hat{\mathbb{C}}$. Then $\Omega_1 \cap \Omega_2$ is a 2-connected domain whose complement $(\Omega_1 \cap \Omega_2)^{\complement}$ has closed connected components $\Omega_1^{\complement}$ and $\Omega_2^{\complement}$.*

Proof. Under the assumption of the lemma suppose that $\Omega_1 \cap \Omega_2$ is a disconnected set. Since the sets Ω_1 and Ω_2 are open and connected, we deduce from Theorem B that the set $\Omega_1 \cup \Omega_2$ separates a certain pair of points $p, q \in \hat{\mathbb{C}}$. This is impossible, because $\Omega_1 \cup \Omega_2 = \hat{\mathbb{C}}$. Therefore $\Omega_1 \cap \Omega_2$ is a connected set. Hence $\Omega_1 \cap \Omega_2$ is a domain, because Ω_1 and Ω_2 are open sets. Note that the complement sets $\Omega_1^{\complement}$ and $\Omega_2^{\complement}$ are closed and
$$\Omega_1^{\complement} \cap \Omega_2^{\complement} = (\Omega_1 \cup \Omega_2)^{\complement} = \emptyset.$$
Since $E(\hat{\mathbb{C}})$ is a normal space, there exist open sets U_1 and U_2 such that
$$0 \in \Omega_1^{\complement} \subset U_1, \ \ \infty \in \Omega_2^{\complement} \subset U_2 \text{ and } U_1 \cap U_2 = \emptyset.$$
Therefore $\Omega_1^{\complement} \cup \Omega_2^{\complement}$ is a disconnected set. Since Ω_1 and Ω_2 are simply connected domains, $\Omega_1^{\complement}$ and $\Omega_2^{\complement}$ are connected sets. Moreover, $(\Omega_1 \cap \Omega_2)^{\complement} = \Omega_1^{\complement} \cup \Omega_2^{\complement}$. Thus the set $(\Omega_1 \cap \Omega_2)^{\complement}$ has exactly two connected components $\Omega_1^{\complement}$ and $\Omega_2^{\complement}$, which proves the lemma. $\square$

From Lemma 2.3 we derive the following result.

Lemma 2.4. *For all simply connected domains $A, B \subset \mathbb{C}$, if $0 \in A \cap B$, then for every $z \in A * B \setminus \{0\}$ the set $A \cap z/B$ is a 2-connected domain as well as the sets $A \setminus (z/B)$ and $A^{\complement}$ are closed components of the set $(A \cap (z/B))^{\complement}$ and $0 \in A \setminus (z/B)$.*

Proof. Fix simply connected domains $A, B \subset \mathbb{C}$ such that $0 \in A \cap B$. Then for a given $z \in A * B \setminus \{0\}$ we conclude from Theorem 2.1 that $A \cup z/B = \hat{\mathbb{C}}$. Moreover, the set z/B is a simply connected domain as well as $\infty \in A^{\complement}$

and $0 \in (z/B)^{\complement}$. Using Lemma 2.3 with $\Omega_1 := A$ and $\Omega_2 := z/B$, we see that $A \cap z/B$ is a 2-connected domain and the set $(A \cap (z/B))^{\complement}$ has closed connected components $A^{\complement}$ and $(z/B)^{\complement}$. Moreover, $0 \in A \setminus (z/B)$ and

$$(z/B)^{\complement} = \hat{\mathbb{C}} \cap (z/B)^{\complement} = (A \cup z/B) \cap (z/B)^{\complement} = A \cap (z/B)^{\complement} = A \setminus (z/B),$$

which is our claim. □

Lemma 2.5. *For all open sets A and B, if $0 \in A \cap B$, then the set $A * B$ is open.*

Proof. Given open sets $A, B \subset \hat{\mathbb{C}}$ assume that $0 \in A \cap B$. Let z be an accumulation point of $A^{\complement} \cdot B^{\complement}$. Then $z_n \to z$ as $n \to +\infty$ for a certain sequence $\mathbb{N} \ni n \mapsto z_n \in A^{\complement} \cdot B^{\complement}$. Hence there exist sequences $\mathbb{N} \ni n \mapsto a_n \in A^{\complement}$ and $\mathbb{N} \ni n \mapsto b_n \in B^{\complement}$ such that $z_n = a_n b_n$ for $n \in \mathbb{N}$. Since $A^{\complement}$ and $B^{\complement}$ are compact sets, there exist $a \in A^{\complement}$ and $b \in B^{\complement}$ and an increasing function $\sigma : \mathbb{N} \to \mathbb{N}$, such that $a_{\sigma(n)} \to a$ and $b_{\sigma(n)} \to b$ as $n \to +\infty$. Since $a \neq 0$ and $b \neq 0$, we have

$$z = \lim_{n\to+\infty} z_{\sigma(n)} = \lim_{n\to+\infty} a_{\sigma(n)} b_{\sigma(n)} = ab \in A^{\complement} \cdot B^{\complement}.$$

Thus $A^{\complement} \cdot B^{\complement}$ is a closed set, and hence $A * B$ is an open set. □

Lemma 2.6. *For all nonempty sets $\Omega \subset \hat{\mathbb{C}}$ and $K \subset \mathbb{C} \setminus \{0\}$, if Ω is open and K is closed, then the set*

$$\Omega_K := \left\{ z \in \mathbb{C} \setminus \{0\} : K \subset \frac{z}{\Omega} \right\} \tag{2.3}$$

is open.

Proof. For arbitrarily fixed non-empty sets $\Omega \subset \mathbb{C}$ and $K \subset \mathbb{C} \setminus \{0\}$ we will prove that the set $\Omega_K^{\complement}$ is closed. From the formula (2.3) it follows that

$$\Omega_K^{\complement} = \left\{ z \in \mathbb{C} \setminus \{0\} : K \cap \frac{z}{\Omega^{\complement}} \neq \emptyset \right\} \cup \{0, \infty\}. \tag{2.4}$$

If $\Omega_K^{\complement} = \{0, \infty\}$, then obviously $\Omega_K^{\complement}$ is closed. Therefore we can assume that $\Omega_K^{\complement} \neq \{0, \infty\}$. Then $\Omega_K^{\complement} \setminus \{0, \infty\} \neq \emptyset$. Suppose that z is arbitrarily fixed adjacent point of the set $\Omega_K^{\complement}$. If $z \in \{0, \infty\}$, then by (2.4), $z \in \Omega_K^{\complement}$. Thus we can constrain ourselves to the case where $z \in \mathbb{C} \setminus \{0\}$. Then there exists a sequence $\mathbb{N} \ni n \mapsto z_n \in \Omega_K^{\complement} \setminus \{0, \infty\}$ such that $z_n \to z$ as $n \to +\infty$. By (2.4) there exists a sequence

$$\mathbb{N} \ni n \mapsto w_n \in K \cap \frac{z_n}{\Omega^{\complement}}. \tag{2.5}$$

Since K is a closed set in a the compact space $\mathrm{E}(\hat{\mathbb{C}})$, there exists an increasing sequence $\sigma : \mathbb{N} \to \mathbb{N}$ and $w \in K$ such that $w_{\sigma(n)} \to w$ as $n \to +\infty$. Since

$K \cap \{0, \infty\} = \emptyset$, we see that $w \in \mathbb{C} \setminus \{0\}$ and $w_n \in \mathbb{C} \setminus \{0\}$ for $n \in \mathbb{N}$, and consequently

$$\frac{z_{\sigma(n)}}{w_{\sigma(n)}} \to \frac{z}{w} \text{ as } n \to +\infty. \tag{2.6}$$

On the other hand, by (2.5), $z_n/w_n \in \Omega^\complement$ for $n \in \mathbb{N}$. Moreover, $\Omega^\complement$ is closed, because Ω is open. This together with (2.6) gives $z/w \in \Omega^\complement$ and, in consequence, $w \in z/\Omega^\complement$. Since $w \in K$, we get $w \in K \cap z/\Omega^\complement$. Hence and by (2.4), $z \in \Omega_K^\complement \setminus \{0, \infty\}$. Thus any adjacent point z of the set $\Omega_K^\complement$ belongs to $\Omega_K^\complement$, and consequently $\Omega_K^\complement$ is a closed set. Hence Ω_K is an open set, which is the desired conclusion. □

Lemma 2.7. *For any compact sets $E, F \subset \mathbb{C}$, $E \cdot F$ is a compact set. If additionally $0 \notin F$, then E/F is a compact set.*

Proof. Fix compact sets $E, F \subset \mathbb{C}$. For any sequence $\mathbb{N} \ni n \mapsto z_n \in E \cdot F$ there exist sequences $\mathbb{N} \ni n \mapsto u_n \in E$ and $\mathbb{N} \ni n \mapsto v_n \in F$ such that $z_n = u_n v_n$ for $n \in \mathbb{N}$. Since E and F are compact, there exist an increasing function $\sigma : \mathbb{N} \to \mathbb{N}$, $u \in E$ and $v \in F$ such that $u_{\sigma(n)} \to u$ and $v_{\sigma(n)} \to v$ as $n \to +\infty$. Hence

$$z_{\sigma(n)} = u_{\sigma(n)} \cdot v_{\sigma(n)} \to u \cdot v \in E \cdot F \text{ as } n \to +\infty,$$

and in a consequence $E \cdot F$ is compact. Now assume that $0 \notin F$. Then the set $1/F$ is compact as the image of the compact set F by a continuous function. Since $E/F = E \cdot 1/F$, we see from the already proved part of the lemma that E/F is a compact set, which completes the proof. □

3. Main results. For any set $\Omega \subset \mathbb{C}$ we define the class $\mathcal{P}(\Omega)$ of all piecewise smooth functions $\gamma : [0; 2\pi] \to \Omega$ satisfying $\gamma(0) = \gamma(2\pi)$, i.e., $\mathcal{P}(\Omega)$ is the class of all closed paths in Ω. Within the class $\mathcal{P}(\Omega)$ we distinguish the subclass

$$\mathcal{P}_0(\Omega) := \{\gamma \in \mathcal{P}(\Omega \setminus \{0\}) : \operatorname{ind}_\gamma(0) = 1\}, \tag{3.1}$$

where $\operatorname{ind}_\gamma(a)$ denotes the index of γ with respect to $a \in \mathbb{C}$.

Theorem 3.1. *Let $A, B \subset \mathbb{C}$ be simply connected domains such that $0 \in A \cap B$. Then $A * B$ is an open set and there exists an operator $T : \mathrm{Hol}(A) \times \mathrm{Hol}(B) \to \mathrm{Hol}(A * B)$ such that for all $f \in \mathrm{Hol}(A)$ and $g \in \mathrm{Hol}(B)$,*

$$T(f, g)(z) = \frac{1}{2\pi \mathrm{i}} \int_\gamma f(u) g\left(\frac{z}{u}\right) \frac{\mathrm{d}u}{u}, \quad z \in A * B \setminus \{0\},\ \gamma \in \mathcal{P}_0\Big(A \cap \frac{z}{B}\Big). \tag{3.2}$$

In particular, the operator T satisfies the conditions (1.10) *and* (1.11), *i.e., $A * B \in \mathcal{H}_2(A, B)$.*

Proof. Fix A, B, f, g satisfying the assumptions and $z \in \Omega \setminus \{0\}$, where $\Omega := A * B$. By Lemma 2.5, Ω is an open set. Since B is simply connected, it follows that z/B is simply connected. Moreover $\infty \in A^{\complement}$, $0 \in (z/B)^{\complement}$ and $A \cup (z/B) = \hat{\mathbb{C}}$, by Theorem 2.1. Using Lemma 2.4 we see that $A \cap (z/B)$ is a 2-connected domain and the set $(A \cap (z/B))^{\complement}$ has closed components $A \setminus (z/B)$ and $A^{\complement}$ containing 0 and ∞, respectively. Since $0 \in B$, we see that $\mathbb{D}(0, 2r) \subset B$ for a certain $r > 0$, and so

$$\mathbb{T}\left(0, \frac{|z|}{r}\right) \subset \frac{z}{B}.$$

If $A = \mathbb{C}$ then

$$\mathbb{T}\left(0, \frac{|z|}{r}\right) \subset A \cap \frac{z}{B},$$

and setting $\Omega_\zeta := A \cap (\zeta/B)$ for $\zeta \in \mathbb{C} \setminus \{0\}$ and

$$[0; 2\pi] \ni t \mapsto \sigma(t) := \frac{|z|}{r}\mathrm{e}^{\mathrm{i}t} \tag{3.3}$$

we see that $\sigma \in \mathcal{P}_0(\Omega_z)$. Otherwise, $A \subset \mathbb{C} \neq A$. Then by the Riemann mapping theorem there exists a conformal mapping Φ of $\mathbb{D}$ onto the domain A such that $\Phi(0) = 0$; cf. [10, Theorem 14.8]. The set $A \setminus (z/B)$ is a compact set as a closed set in the compact space $E(\hat{\mathbb{C}})$. Since

$$\bigcup_{n\in\mathbb{N}} \Phi\Big(\mathbb{D}\Big(0, 1 - \frac{1}{n}\Big)\Big) = \Phi\Big(\bigcup_{n\in\mathbb{N}} \mathbb{D}\Big(0, 1 - \frac{1}{n}\Big)\Big) = \Phi(\mathbb{D}) = A \supset A \setminus \Omega_z,$$

the family $\big\{\Phi(\mathbb{D}(0, 1 - 1/n)) : n \in \mathbb{N}\big\}$ is an open cover of $A \setminus \Omega_z$. Therefore there exists a finite set $N \subset \mathbb{N}$ such that

$$A \setminus \Omega_z \subset \bigcup_{n\in N} \Phi\Big(\mathbb{D}\Big(0, 1 - \frac{1}{n}\Big)\Big) = \Phi\Big(\mathbb{D}\Big(0, 1 - \frac{1}{n_z}\Big)\Big),$$

where $n_z := \max(N)$. Setting $r := 1 - 1/(2n_z)$ we see that

$$\begin{aligned}\Phi\left(\mathbb{T}(0, r)\right) \subset \Phi\left(\mathbb{D} \setminus \mathbb{D}\left(0, 1 - \frac{1}{n_z}\right)\right) &= \Phi(\mathbb{D}) \setminus \Phi\left(\mathbb{D}\left(0, 1 - \frac{1}{n_z}\right)\right)\\ &\subset A \setminus (A \setminus \Omega_z) = \Omega_z.\end{aligned}$$

Hence $\sigma \in \mathcal{P}(\Omega_z)$, where

$$[0; 2\pi] \ni t \mapsto \sigma(t) := \Phi\left(r\mathrm{e}^{\mathrm{i}t}\right). \tag{3.4}$$

Since Φ is a conformal mapping, we conclude from the residue theorem that

$$\mathrm{ind}_\sigma(0) = \frac{1}{2\pi\mathrm{i}} \int_\sigma \frac{\mathrm{d}u}{u} = \frac{1}{2\pi\mathrm{i}} \int_{\mathbb{T}^+(0,r)} \frac{\Phi'(w)}{\Phi(w)} \mathrm{d}w = 1.$$

Hence $\sigma \in \mathcal{P}_0(\Omega_z)$, and so in both the cases ($A = \mathbb{C}$ or $A \neq \mathbb{C}$),

$$\mathcal{P}_0(\Omega_z) \neq \emptyset, \quad z \in \Omega \setminus \{0\}. \tag{3.5}$$

Now we consider the following function

$$\bigcup_{w\in\Omega\setminus\{0\}} \{w\} \times \mathcal{P}_0(\Omega_w) \ni (z,\gamma) \mapsto h(z,\gamma) := \frac{1}{2\pi\mathrm{i}} \int_\gamma H_z(u)\mathrm{d}u, \tag{3.6}$$

where

$$\Omega_z \ni u \mapsto H_z(u) := f(u) g\left(\frac{z}{u}\right)\frac{1}{u}$$

is a well defined holomorphic function in Ω_z. For any fixed $\gamma_1, \gamma_2 \in \mathcal{P}_0(\Omega_z)$, $\Gamma := \{(\gamma_1, 1), (\gamma_2, -1)\}$ is a cycle in Ω_z such that for every $a \in A \setminus \Omega_z$,

$$\mathrm{ind}_\Gamma(a) = \mathrm{ind}_{\gamma_1}(a) - \mathrm{ind}_{\gamma_2}(a) = \mathrm{ind}_{\gamma_1}(0) - \mathrm{ind}_{\gamma_2}(0) = 1 - 1 = 0$$

and for every $a \in \mathbb{C} \setminus A$,

$$\mathrm{ind}_\Gamma(a) = \mathrm{ind}_{\gamma_1}(a) - \mathrm{ind}_{\gamma_2}(a) = 0 - 0 = 0.$$

Hence Γ is a cycle homologous to zero with respect to Ω_z. Using the homologous version of Cauchy theorem we conclude that

$$0 = \int_\Gamma H_z(u)\mathrm{d}u = \int_{\gamma_1} H_z(u)\mathrm{d}u - \int_{\gamma_2} H_z(u)\mathrm{d}u,$$

which together with (3.6) leads to

$$h(z,\gamma_1) = \frac{1}{2\pi\mathrm{i}} \int_{\gamma_1} H_z(u)\mathrm{d}u = \frac{1}{2\pi\mathrm{i}} \int_{\gamma_2} H_z(u)\mathrm{d}u = h(z,\gamma_2).$$

Therefore the function h is constant with respect to the second argument. From (3.5) we conlude that there exists a function $h : \Omega \to \mathbb{C}$ satisfying the following conditions

$$h(z) = h(z,\gamma), \quad z \in \Omega \setminus \{0\},\ \gamma \in \mathcal{P}_0(\Omega_z) \tag{3.7}$$

and

$$h(0) = f * g(0). \tag{3.8}$$

Now we prove that $h \in \mathrm{Hol}(\Omega \setminus \{0\})$. Setting $K := \sigma([0; 2\pi])$ we see that $K \subset \mathbb{C} \setminus \{0\}$ and K is closed. Since B is an open set, we conclude from Lemma 2.6 that the set

$$B_K := \left\{\zeta \in \mathbb{C} \setminus \{0\} : K \subset \frac{\zeta}{B}\right\} \tag{3.9}$$

is open. Since $\sigma \in \mathcal{P}_0(\Omega_z)$, we have $K \subset \Omega_z = A \cap z/B$, and so $K \subset z/B$. Then by (3.9), $z \in B_K$. Thus $\mathbb{D}(z,\eta) \subset B_K$ for a certain $\eta > 0$. This means that for each $w \in \mathbb{D}(z,\eta)$, $w \in B_K$, which in view of (3.9) gives $K \subset (w/B) \cap A = \Omega_w$. Thus

$$\sigma \in \mathcal{P}_0(\Omega_w), \quad w \in \mathbb{D}(z,\eta). \tag{3.10}$$

Notice that by (3.7) and (3.6),

$$\begin{aligned} h(w) &= \frac{1}{2\pi \mathrm{i}} \int_\sigma f(u) g\left(\frac{w}{u}\right) \frac{\mathrm{d}u}{u} \\ &= \frac{1}{2\pi \mathrm{i}} \int_0^{2\pi} f(\sigma(t)) g\left(\frac{w}{\sigma(t)}\right) \frac{\sigma'(t)}{\sigma(t)} \mathrm{d}t, \ w \in \mathbb{D}(z,\eta). \end{aligned}$$

Since the function under the last integral is holomorphic with respect to w, we see that the function h is differentiable at the point z and

$$h'(z) = \frac{1}{2\pi \mathrm{i}} \int_0^{2\pi} f(\sigma(t)) g'\left(\frac{z}{\sigma(t)}\right) \frac{\sigma'(t)}{\sigma(t)^2} \mathrm{d}t = \frac{1}{2\pi \mathrm{i}} \int_\sigma f(u) g'\left(\frac{z}{u}\right) \frac{\mathrm{d}u}{u^2};$$

cf. e.g. [5, Chapter 6, Theorem 5.4]. Thus

$$h \in \mathrm{Hol}(\Omega \setminus \{0\}). \tag{3.11}$$

By the formula (1.3), $\mathbb{D}(0,\rho(A)) \subset A$ and $\mathbb{D}(0,\rho(B)) \subset B$. Hence and by (1.8),

$$\mathbb{D}(0,\rho(A)\rho(B)) \subset \Omega. \tag{3.12}$$

We still have to prove that

$$h(z) = f * g(z), \ z \in \mathbb{D}(0,\rho(A)\rho(B)). \tag{3.13}$$

Since $0 \in A \cap B \cap \Omega$ and the sets A, B, Ω are open, there exists $r \in (0;1)$ such that $\mathbb{D}(0,r) \subset \Omega \cap A \cap B$. Hence $f, g \in \mathrm{Hol}(\mathbb{D}(0,r))$, and consequently by Theorem A,

$$f * g(z) = \frac{1}{2\pi \mathrm{i}} \int_\gamma f(u) g\Big(\frac{z}{u}\Big) \frac{\mathrm{d}u}{u}, \ z \in \mathbb{D}\Big(0, \frac{r^2}{2}\Big), \tag{3.14}$$

where $[0;2\pi] \ni t \mapsto \gamma(t) := (r/2)\mathrm{e}^{\mathrm{i}t}$. For every $z \in \mathbb{D}(0,r^2/2) \setminus \{0\}$,

$$\gamma([0;2\pi]) = \mathbb{T}(0,r/2) \subset \mathbb{C} \setminus \overline{\mathbb{D}}(0,|z|/r) \subset z/B,$$

because $\mathbb{D}(0,r) \subset B$ and $|z|/r < r/2$. Moreover, $\gamma([0;2\pi]) \subset \mathbb{D}(0,r) \subset A$ and $\mathrm{ind}_\gamma(0) = 1$. Therefore $\gamma \in \mathcal{P}_0(A \cap z/B) = \mathcal{P}_0(\Omega_z)$, and using (3.7) and (3.6) we obtain

$$h(z) = h(z,\gamma) = \frac{1}{2\pi \mathrm{i}} \int_\gamma f(u) g\Big(\frac{z}{u}\Big) \frac{\mathrm{d}u}{u}, \ z \in \mathbb{D}\Big(0, \frac{r^2}{2}\Big) \setminus \{0\}.$$

This together with (3.14) gives

$$f * g(z) = h(z), \ z \in \mathbb{D}\Big(0, \frac{r^2}{2}\Big) \setminus \{0\}. \tag{3.15}$$

Combining this with (3.8) and (3.11) we see that $h \in \mathrm{Hol}(\Omega)$. Moreover, from (3.15), (3.12) and (1.4) it follows that the equality (3.13) holds. Thus an operator $T : \mathrm{Hol}(A) \times \mathrm{Hol}(B) \to \mathrm{Hol}(A * B)$ is well defined by setting $T(f,g) := h$, and the equality (3.2) holds.

It remains to show that the operator T satisfies the conditions (1.10) and (1.11). Write $\mathbb{Z}_{p,q} := \{n \in \mathbb{Z} : p \leq n \leq q\}$ and $\mathbb{Z}_p := \{n \in \mathbb{Z} : p \leq n\}$ for $p, q \in \mathbb{Z}$. For any polynomials f and g there exist $n \in \mathbb{N}$ and sequences $\mathbb{Z}_{0,n} \ni k \mapsto a_k \in \mathbb{C}$ and $\mathbb{Z}_{0,n} \ni k \mapsto b_k \in \mathbb{C}$ such that

$$f(z) = \sum_{k=0}^{n} a_k z^k \text{ and } g(z) = \sum_{k=0}^{n} b_k z^k, \ z \in \mathbb{C}.$$

Then for all $z \in \Omega \setminus \{0\}$ and $\gamma \in \mathcal{P}_0\Big(A \cap \dfrac{z}{B}\Big)$ we conclude from (3.2) that

$$\begin{aligned} T(f,g)(z) &= \frac{1}{2\pi \mathrm{i}} \int_\gamma f(u) g\Big(\frac{z}{u}\Big) \frac{\mathrm{d}u}{u} \\ &= \frac{1}{2\pi \mathrm{i}} \int_\gamma \Big(\sum_{k=0}^{n} a_k u^k\Big)\Big(\sum_{l=0}^{n} b_l \frac{z^l}{u^l}\Big) \frac{\mathrm{d}u}{u} \\ &= \sum_{k=0}^{n} \sum_{l=0}^{n} \frac{a_k b_l z^l}{2\pi \mathrm{i}} \int_\gamma u^{k-l-1} \mathrm{d}u \\ &= \sum_{k=0}^{n} a_k b_k z^k = f * g(z). \end{aligned}$$

This shows the condition (1.10). Given $f \in \mathrm{Hol}(A)$, $g \in \mathrm{Hol}(B)$ and sequences $\mathbb{N} \ni n \mapsto f_n \in \mathrm{Hol}(A)$ and $\mathbb{N} \ni n \mapsto g_n \in \mathrm{Hol}(B)$ suppose that

$$f_n \xrightarrow{\text{ucc}} f \text{ in } A \text{ and } g_n \xrightarrow{\text{ucc}} g \text{ in } B \text{ as } n \to +\infty. \tag{3.16}$$

Let $z \in \Omega \setminus \{0\}$ be arbitrarily fixed. We have already proved that there exists $\gamma_z \in \mathcal{P}_0(\Omega_z)$. Taking into account (3.10) we see that $\mathbb{D}(z, 2r_z) \subset \Omega \setminus \{0\}$ and

$$\gamma_z \in \mathcal{P}_0(\Omega_w), \ w \in \mathbb{D}(z, 2r_z), \tag{3.17}$$

for a certain $r_z > 0$. By (3.17), $K_z := \gamma_z([0; 2\pi]) \subset \Omega_w = A \cap w/B$ for $w \in \mathbb{D}(z, 2r_z)$. Hence $K'_z := \overline{\mathbb{D}}(z, r_z)/K_z \subset B$, and in view of Lemma 2.7 the set K'_z is compact. Since both the sets K_z and K'_z are compact, we conclude from (3.16) that

$$\sup_{u \in K_z} |f_n(u) - f(u)| \to 0 \text{ and } \sup_{u \in K'_z} |g_n(u) - g(u)| \to 0 \text{ as } n \to +\infty.$$

Therefore, for a given $\varepsilon \in (0; 1)$ there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\sup_{u \in K_z} |f_n(u) - f(u)| < \frac{\varepsilon}{M_z + 1} \text{ and } \sup_{u \in K'_z} |g_n(u) - g(u)| < \frac{\varepsilon}{M_z + 1} \tag{3.18}$$

for every $n \in \mathbb{Z}_{n_\varepsilon}$, where

$$M_z := \max\Big(\Big\{\sup_{u \in K_z} |f(u)|, \sup_{u \in K'_z} |g(u)|\Big\}\Big) < +\infty.$$

Setting $d_z := \inf_{u\in K_z} |u|$ we have $d_z > 0$. Using now (3.2) we conclude from (3.18) that for all $w \in \mathbb{D}(z, r_z)$ and $n \in \mathbb{Z}_{n_\varepsilon}$,

$$|T(f_n, g_n)(w) - T(f_n, g)(w)| \le \frac{1}{2\pi} \int_{\gamma_z} |f_n(u)| \Big| g_n\Big(\frac{w}{u}\Big) - g\Big(\frac{w}{u}\Big)\Big| \frac{|\mathrm{d}u|}{|u|} \le \frac{|\gamma_z|_1}{2\pi d_z}\varepsilon$$

as well as

$$|T(f_n, g)(w) - T(f, g)(w)| \le \frac{1}{2\pi} \int_{\gamma_z} |f_n(u) - f(u)| \Big| g\Big(\frac{w}{u}\Big)\Big| \frac{|\mathrm{d}u|}{|u|} \le \frac{|\gamma_z|_1}{2\pi d_z}\varepsilon.$$

Thus

$$\sup_{w\in\mathbb{D}(z,r_z)} |T(f_n, g_n)(w) - T(f, g)(w)| \to 0 \text{ as } n \to +\infty. \tag{3.19}$$

Suppose that E is a non-empty compact set such that $E \subset \Omega \setminus \{0\}$. Since $E \subset \bigcup_{z\in E} \mathbb{D}(z, r_z)$, there exists a finite subset $E' \subset E$ such that $E \subset \bigcup_{z\in E'} \mathbb{D}(z, r_z)$. Fix $\varepsilon > 0$. By (3.19) for each $z \in E'$ there exists $n_z \in \mathbb{N}$ such that

$$\Big|T(f_n, g_n)(w) - T(f, g)(w)\Big| < \varepsilon, \quad n \in \mathbb{Z}_{n_z},\ w \in \mathbb{D}(z, r_z). \tag{3.20}$$

Setting $N_\varepsilon := \max\big(\{n_z : z \in E'\}\big)$ we conclude from (3.20) that

$$|T(f_n, g_n)(w) - T(f, g)(w)| < \varepsilon, \quad n \in \mathbb{Z}_{N_\varepsilon},\ w \in E,$$

and so the sequence $\mathbb{N} \ni n \mapsto T(f_n, g_n)$ is uniformly convergent to $T(f, g)$ in E. Therefore

$$T(f_n, g_n) \xrightarrow{\text{ucc}} T(f, g) \text{ in } \Omega \setminus \{0\} \text{ as } n \to +\infty. \tag{3.21}$$

Since Ω is open and $0 \in \Omega$, there exists $r > 0$ with the property $\overline{\mathbb{D}}(0, r) \subset \Omega$. Hence $\mathbb{T}(0, r)$ is a compact subset of $\Omega \setminus \{0\}$. Then using the maximum principle for holomorphic functions we deduce from (3.21) that

$$\begin{aligned} &\sup_{w\in\overline{\mathbb{D}}(0,r)} |T(f_n, g_n)(w) - T(f, g)(w)| \\ &\qquad = \sup_{w\in\mathbb{T}(0,r)} |T(f_n, g_n)(w) - T(f, g)(w)| \to 0 \text{ as } n \to +\infty. \end{aligned} \tag{3.22}$$

Notice that for every compact set $E \subset \Omega$, $E \subset \overline{\mathbb{D}}(0, r) \cup (E \setminus \mathbb{D}(0, r))$ and $E \setminus \mathbb{D}(0, r)$ is a compact subset of $\Omega \setminus \{0\}$. Combining this with (3.21) and (3.22) we obtain

$$T(f_n, g_n) \xrightarrow{\text{ucc}} T(f, g) \text{ in } \Omega \text{ as } n \to +\infty.$$

This shows the condition (1.11), which completes the proof. □

Corollary 3.2. *For all simply connected domains $A, B \subset \mathbb{C}$, if $0 \in A \cap B$, then $\mathrm{CC}_0(A * B) \in \mathcal{H}_1(A, B)$.*

Proof. Fix A and B satisfying the assumption. By Lemma 2.5, $\Omega := \mathrm{CC}_0(A*B)$ is a domain. From Theorem 3.1 it follows that for arbitrarily fixed $f \in \mathrm{Hol}(A)$ and $g \in \mathrm{Hol}(B)$, $h := T(f,g)|_\Omega \in \mathrm{Hol}(\Omega)$. By (3.12) we have $\mathbb{D}(0,\rho(A)\rho(B)) \subset \Omega$. Since $f*g \in \mathrm{Hol}(\mathbb{D}(0,R_{f,g}))$, we deduce from (1.4) that $f*g \in \mathrm{Hol}(\mathbb{D}(0,\rho(A)\rho(B))$. Notice that $f_m \xrightarrow{\text{ucc}} f$ in $\mathbb{D}(0,\rho(A))$ and $g_m \xrightarrow{\text{ucc}} g$ in $\mathbb{D}(0,\rho(B))$ as $m \to +\infty$, where

$$\mathbb{C} \ni z \mapsto f_m(z) := \sum_{n=0}^{m} \frac{f^{(n)}(0)}{n!} z^n \text{ and } \mathbb{C} \ni z \mapsto g_m(z) := \sum_{n=0}^{m} \frac{g^{(n)}(0)}{n!} z^n, \; m \in \mathbb{N}.$$

By (1.1), $f_m * g_m \xrightarrow{\text{ucc}} f*g$ in $\mathbb{D}(0,\rho(A)\rho(B))$ as $m \to +\infty$. On the other hand the operator T satisfies the conditions (1.10) and (1.11), from which

$$f_m * g_m = T(f_m, g_m) \xrightarrow{\text{ucc}} T(f,g) \text{ in } \mathbb{D}(0,\rho(A)\rho(B)) \text{ as } m \to +\infty.$$

Therefore $f*g(z) = h(z)$ for $z \in \mathbb{D}(0,\rho(A)\rho(B))$, i.e., the condition (1.5) holds. Then in view of the definition of the class $\mathcal{H}_1(A,B)$ we obtain $\Omega \in \mathcal{H}_1(A,B)$, which is the desired conclusion. □

Remark 3.3. Under the assumption of Corollary 3.2 we conclude from (1.9) and the corollary that $\Omega := \mathrm{CC}_0(A*B)$ is the greatest domain in the class $\mathcal{H}_1(A,B)$ in the sense of the inclusion. If additionally $A*B$ is connected, then $A*B = \mathrm{CC}_0(A*B)$, and so $A*B \in \mathcal{H}_1(A,B)$. Moreover, $h := T(f,g)|_\Omega$ is the unique holomorphic function satisfying the condition (1.5) and, by the formula (3.2), the function h has an explicit integral representation.

Let us notice that in Corollary 3.2 the set $\mathrm{CC}_0(A*B)$ can not be replaced by the whole set $A*B$ in general. This is illustrated by the following example.

Example 3.4. Setting

$$A := B := \mathbb{C} \setminus \left([1;+\infty) \cup \left\{e^{it} : t \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]\right\}\right)$$

we calculate

$$A * B = (\mathbb{D} \cup \{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}) \setminus \mathbb{T}.$$

Thus $A*B$ is not connected although both domains A and B are simply connected.

References

[1] Grosse-Erdmann, K. G., *On the Borel–Okada theorem and the Hadamard multiplication theorem*, Complex Variables Theory Appl. **22** (1993), 101–112.

[2] Hadamard, J., *Théoreme sur les series entieres*, Acta Math. **22** (1899), 55–63 (French).

[3] Hille, E., *Analytic Function Theory, Vol. II*, Chelsea Publishing Company, New York, 1959.

[4] Kuratowski, K., *Topology, Vol II*, PWN, Warszawa, 1968.

[5] Levinson, N., Redheffer, R. M., *Complex Variables*, Holden-Day, Inc., San Francisco, Calif.–Cambridge–Amsterdam, 1970.

[6] Lorson, T., *Hadamard Convolution Operators on Spaces of Holomorphic Functions*, Dissertation, University of Trier, 2013.

[7] Lorson, T., Müller, J., *Convolution operators on spaces of holomorphic functions*, Studia Math. **227** (2015), 111–131.

[8] Müller, J., *The Hadamard multiplication theorem and applications in summability theory*, Complex Variables Theory Appl. **18** (1992), 155–166.

[9] Müller, J., Pohlen, T., *The Hadamard product on open sets in the extended plane*, Complex Anal. Oper. Theor. **6** (2012), 257–274.

[10] Rudin, W., *Real and Complex Analysis*, third ed., McGraw-Hill International Editions, Mathematics Series, McGraw-Hill Book Company, Singapore, 1987.

Maciej Parol
Department of Mathematical Analysis
The John Paul II Catholic University of Lublin
Al. Racławickie 14, P.O. Box 129
20-950 Lublin
Poland
e-mail: `mparol@kul.lublin.pl`

Dariusz Partyka
Department of Mathematical Analysis
The John Paul II Catholic University of Lublin
Al. Racławickie 14, P.O. Box 129
20-950 Lublin
Poland
e-mail: `partyka@kul.lublin.pl`

Institute of Mathematics and Information Technology
The State School of Higher Education in Chełm
ul. Pocztowa 54
22-100 Chełm
Poland

Received November 22, 2021