doi: 10.17951/a.2022.76.1.61-71

ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LXXVI, NO. 1, 2022 SECTIO A 61–71

NERMIN OKIČIĆ, AMRA REKIĆ-VUKOVIĆ and VEDAD PAŠIĆ

A new characterization of strict convexity on normed linear spaces

Abstract. We consider relations between the distance of a set A and the distance of its translated set $A+x$ from 0, for $x \in A$, in a normed linear space. If the relation $d(0, A+x) < d(0, A)+\|x\|$ holds for exactly determined vectors $x \in A$, where A is a convex, closed set with positive distance from 0, which we call (TP) property, then this property is equivalent to strict convexity of the space. We show that in uniformly convex spaces the considered property holds.

1. Introduction. Translation of the set as a simple transformation was not often considered in normed spaces. The distance of the set and of its translated set from 0 in the space show some regularities. Properties of the considered space determine the behavior of the mentioned distances for convex and closed sets in the given normed space. In the main part of this paper we will show that using this property, we obtain a characterization of strict convexity of the normed space, but also that uniform convexity determines the relation between the distance of the set and its translated set from 0 in the considered normed space.

The metric space in which we have a vector structure is called a linear metric space. If the metric is obtained from the norm, we call such space a normed linear space. In what follows we denote by X the normed linear space. $B_X = \{x \in X \mid d(0, x) \leq 1\}$ and $S_X = \{x \in X \mid d(0, x) = 1\}$ denote

2010 *Mathematics Subject Classification.* 46B20, 52A27.

Key words and phrases. Translation, uniform convexity, strict convexity.

the unit ball and the unit sphere in the given space, where the metric is induced by the norm of the space, $d(x,y) = \|x-y\|$. A set A in X is said to be convex if, for all x and y in A and all $\lambda \in [0,1]$, the point $\lambda x + (1-\lambda)y$ also belongs to A. By $\operatorname{conv} A$ and $\overline{\operatorname{conv}} A$ we denote the convex hull and the closure of the convex hull of the set A respectively.

2. Some translation properties. The translation of the set $A \subset X$ by the vector $x \in X$ is the set

$$A + x = \{y + x \mid y \in A\} .$$

Let A be a nonempty subset of a normed space X. For every $x \in X$, the distance between the point x and the set A is denoted by $d(x,A)$ and is defined by the following formula

$$d(x,A) = \inf\{d(x,y) \mid y \in A\} .$$

Lemma 2.1. *Let X be a normed linear space and $A \subset X$ be a convex, closed set with $d(0,A) > 0$. Then, for all $x \in A$ we have $d(0,A+x) > 0$.*

Proof. Suppose that there exists $x_0 \in A$ with $d(0,A+x_0) = 0$. Let $(x_n + x_0)_{n\in\mathbb{N}}$ be the minimizing sequence such that

$$d(0, x_n + x_0) = \|x_n + x_0\| \to \ 0\,,\ n \to \infty .$$

We conclude that $x_n + x_0 \to 0$ when $n \to \infty$, i.e.,

$$x_n \to -x_0,\ n \to \infty .$$

Since $(x_n)_{n\in\mathbb{N}} \subset A$ and A is closed, we see that $-x_0 \in A$. So, we have $x_0, -x_0 \in A$ and because of convexity of the set, we conclude that $0 \in A$. But this contradicts the assumption that $d(0,A) > 0$. □

Lemma 2.2. *Let X be a normed linear space and let $A \subset X$ be a convex set with $d(0,A) > 0$. Then for an arbitrary $x \in A$,*

$$d(0, A+x) > d(0,A) .$$

Proof. Suppose that for some $x_0 \in A$ we have

$$d(0, A+x_0) \leq d(0,A) .$$

One can choose minimizing sequences $(x_n)_{n\in\mathbb{N}} \subset A$ and $(y_n)_{n\in\mathbb{N}} \subset A + x_0$, that is

$$\|x_n\| \to d(0,A)\,,\ n \to \infty\,, \tag{1}$$

$$\|y_n\| \to d(0, A+x_0)\,,\ n \to \infty\,, \tag{2}$$

where $y_n = z_n + x_0$ for some $z_n \in A$ ($n \in \mathbb{N}$) and such that the inequality

$$\|z_n + x_0\| \leq \|x_n\| \tag{3}$$

holds for almost all $n \in \mathbb{N}$. Let $\varepsilon < \frac{d(0,A)}{2}$ be arbitrary. From (1) and (2) we get

$$(\exists n_1 \in \mathbb{N})\, (\forall n \in \mathbb{N})\, (n \geq n_1 \ \Rightarrow\ \|x_n\| < d(0,A) + \varepsilon)\,,$$
$$(\exists n_2 \in \mathbb{N})\, (\forall n \in \mathbb{N})\, (n \geq n_2 \ \Rightarrow\ \|z_n + x_0\| < d(0,A + x_0) + \varepsilon)\,.$$

Let $n_0 = \max\{n_1, n_2\}$. Then for all $n \in \mathbb{N}$,

$$n \geq n_0 \ \Rightarrow\ \|x_n\| < d(0,A) + \varepsilon \ \wedge\ \|z_n + x_0\| < d(0,A + x_0) + \varepsilon\,. \tag{4}$$

Since x_0 and z_n belong to the set A which is convex, for an arbitrary $\lambda \in [0,1]$ we have $\lambda x_0 + (1-\lambda) z_n \in A$. In particular, for $\lambda = \frac{1}{2}$, because of (3), the following inequality holds

$$\left\| \frac{1}{2} x_0 + \frac{1}{2} z_n \right\| \leq \frac{1}{2} \|x_n\|\,.$$

Using (4) and the choice of the ε, we get

$$\left\| \frac{1}{2} x_0 + \frac{1}{2} z_n \right\| < \frac{1}{2}(d(0,A) + \varepsilon) < \frac{3}{4} d(0,A)\,.$$

So, $x^* = \dfrac{1}{2} x_0 + \dfrac{1}{2} z_n \in A$ and

$$\|x^*\| < \frac{3}{4} d(0,A) < d(0,A) = \inf\{\|x\| \mid x \in A\}\,,$$

which is an obvious contradiction. □

In the general case, if the distance of the set A from 0 is obtained by the sequence $(x_n)_{n\in\mathbb{N}} \subset A$, i.e.,

$$\|x_n\| \to d(0,A)\,,\ n \to \infty\,,$$

then for $x_0 \in A$ the sequence $(x_n + x_0)_{n\in\mathbb{N}}$ does not have to be minimizing for the distance of the set $A + x_0$ from 0. Indeed, consider the space $\mathbb{R}^2$ with the Euclidean metric and $A \subset \mathbb{R}^2$ given by $A = \{(x,y) \in \mathbb{R}^2 \mid x = 1\,,\ y \in [-1,1]\}$. The set A is clearly convex and closed and equality $d(0,A) = 1 = d((0,0),(1,0))$ holds. Consider then $x_0 = (1,1) \in A$ and translation of the set A by this vector, i.e.,

$$A + x_0 = \{(x,y) \in \mathbb{R}^2 \mid x = 2\,,\ y \in [0,2]\}\,.$$

Then

$$\begin{aligned} d(0, A + x_0) &= \inf\{d((0,0),(x,y) + (1,1)) \mid (x,y) \in A\} \\ &= \inf\{d((0,0),(2,y+1)) \mid y \in [-1,1]\} \\ &= \inf\left\{\sqrt{4 + (y+1)^2} \mid y \in [-1,1]\right\} \\ &= 2 = d((0,0),(2,0))\,. \end{aligned}$$

Since $d(0,A) = d(0,x^*)$, where $x^* = (1,0) \in A$ and since

$$d(0, x^* + x_0) = d((0,0),(2,1)) = \sqrt{5}\,,$$

we have

$$d(0, A + x_0) = 2 < \sqrt{5} = d(0, x^* + x_0)\,.$$

Lemma 2.3. *In the general case of a normed linear space X, for $A \subset X$ and $\alpha \in \mathbb{R}$ we have*

1. $d(x, \alpha A) = |\alpha| d(\alpha^{-1}x, A)$ *for an arbitrary* $x \in X$.
2. $d(x, A) = d(x + y, A + y)$ *for arbitrary* $x, y \in X$.
3. $d(x, x + z) = d(y, y + z)$ *for arbitrary* $x, y, z \in X$.

Lemma 2.4. *Let X be a normed linear space and $A \subset X$ be such that $d(0, A) > 0$. For an arbitrary $x \in A$ it holds*

$$d(0, A + x) \leq d(0, A) + \|x\|\,.$$

Proof. Let $x \in A$ be arbitrary and fixed. Then

$$A + x = \{a + x \mid a \in A\}\,.$$

For an arbitrary $y \in A$ we have

$$d(0, A + x) = \inf\{\|a + x\| \mid a \in A\} \leq \|y + x\| \leq \|y\| + \|x\|\,.$$

Since the left hand side of the above inequality does not depend on y, we see that

$$\begin{aligned} d(0, A + x) &\leq \inf\{\|y\| + \|x\| \mid y \in A\} = \inf\{\|y\| \mid y \in A\} + \|x\| \\ &= d(0, A) + \|x\|\,. \end{aligned}$$ □

Lemma 2.5. *Let X be a normed linear space and let $A \subset X$ be a convex and closed set with $d(0, A) > 0$. If there exists $x^* \in A$ such that $d(0, A) = \|x^*\|$, then for all $t \in [1, +\infty)$ such that $tx^* \in A$, we have*

$$d(0, A + tx^*) = d(0, A) + t\|x^*\|\,.$$

Proof. Let $t \in [1, +\infty)$ be such that $tx^* \in A$. Lemma 2.4 implies

$$d(0, A + tx^*) \leq d(0, A) + \|tx^*\| = d(0, A) + t\|x^*\|\,.$$

Suppose that $d(0, A + tx^*) < d(0, A) + t\|x^*\|$. Then there exists $y \in A$ such that

$$\|y + tx^*\| < d(0, A) + t\|x^*\| = (1 + t)\|x^*\|\,.$$

We conclude that

$$\left\| \frac{1}{1+t} y + \frac{t}{1+t} x^* \right\| < \|x^*\|$$

and since the set A is convex, $\frac{1}{1+t} y + \frac{t}{1+t} x^* \in A$ holds. Hence, the last inequality gives a contradiction with the assumption that $d(0, A) = \|x^*\|$. Therefore

$$d(0, A + tx^*) = d(0, A) + t\|x^*\|\,.$$ □

As a special case of Lemma 2.5, for $t = 1$, there exists $x^* \in A$ such that $d(0, A) = \|x^*\|$ and the following equality holds

$$d(0, A + x^*) = 2\|x^*\| = 2d(0, A) = d(0, A) + \|x^*\|\,.$$

3. Main results. We recall some notions.

The normed space X is called uniformly convex, for short (UC), if for every ε, $0 < \varepsilon \leq 2$, there exists $\delta = \delta(\varepsilon) > 0$ such that for every $x, y \in B_X$,

$$\left\| \frac{x+y}{2} \right\| > 1 - \delta \Rightarrow \|x - y\| < \varepsilon \,,$$

or equivalently

$$\|x - y\| \geq \varepsilon \Rightarrow \left\| \frac{x+y}{2} \right\| \leq 1 - \delta \,.$$

The normed space X is called strictly convex, for short (SC), if its unit sphere S_X does not contain nontrivial segments, that is, for every $x, y \in S_X$, $x \neq y$, $[x, y] \not\subset S_X$. This means that for all $t \in (0, 1)$, we have $\|(1-t)x + ty\| < 1$, or equivalently, if the equality $\|(1-t)x + ty\| = 1$ holds for some $x, y \in S_X$ and some $t \in (0, 1)$, then $x = y$. For more about uniform and strict convexity see [1] and [3].

The following proposition contains some equivalent conditions for strict convexity (see [2]).

Proposition 3.1. *For a normed space X, the following conditions are equivalent:*

(1) X is strictly convex;
(2) for every $x, y \in S_X$ with $x \neq y$, $\|x + y\| < 2$;
(3) for every $x, y \in X \backslash \{0\}$, the equality $\|x + y\| = \|x\| + \|y\|$ implies $y = \alpha x$ for some $\alpha > 0$.

Evidently, every (UC) space is a (SC) space.

Lemma 2.5 holds for the spaces that have the property that for a convex and closed set A, with $d(0, A) > 0$, there exists $x^* \in A$ such that $d(0, A) = \|x^*\|$. For example, Hilbert and reflexive spaces have this property.

Let X be a normed space, $\emptyset \neq Z \subset X$ and $x \in X$. By $P_Z(x)$ we denote the set

$$P_Z(x) = \{z \in Z \mid d(x, Z) = \|x - z\|\} \,.$$

In [2], it was shown that if Z is a convex and closed set and X is a strictly convex space, then the following holds

$$(\forall x \in X \setminus Z) \ \ \mathrm{card}(P_Z(x)) \leq 1 \,.$$

In particular, if X is a uniformly convex space, then

$$(\forall x \in X \setminus Z) \ \ \mathrm{card}(P_Z(x)) = 1 \,.$$

It is clear that Lemma 2.5 holds in (UC) spaces, but it is also clear that this assertion can not be applied to (SC) spaces because we do not know if there exists $x^* \in Z$ such that $d(0, Z) = \|x^*\|$.

Let us prove a few auxiliary claims.

Lemma 3.2. *Let X be a normed linear space and $x, y \in X$ be nonzero vectors such that $\|x+y\| = \|x\| + \|y\|$. Then for vectors $x_0 = \frac{x}{\|x\|}$, $y_0 = \frac{y}{\|y\|} \in S_X$ the equality $\|x_0 + y_0\| = 2$ holds.*

Proof. Let $x \in X$ be an arbitrary nonzero vector and $x_0 = \frac{x}{\|x\|}$. Consider the set $A = \{\lambda x \mid \lambda \in (0, +\infty)\}$. Suppose that $A \cap S_X = \{x_1, x_2\}$. Because $x_1, x_2 \in A$, there exist $\lambda_1, \lambda_2 \in (0, +\infty)$ such that $x_1 = \lambda_1 x$ and $x_2 = \lambda_2 x$. Because $x_1, x_2 \in S_X$, we have $\|\lambda_1 x\| = \|\lambda_2 x\| = 1$, so we conclude that $\lambda_1 = \lambda_2$, that is $x_1 = x_2$. Obviously, for $\lambda = \frac{1}{\|x\|}$, $x_0 \in A$ and $\|x_0\| = 1$, that is $x_0 \in S_X$. Therefore, $A \cap S_X = \{x_0\}$.

Let $x, y \in X$ be nonzero vectors such that $\|x+y\| = \|x\| + \|y\|$. Without lost of generality let $\|x\| > 1$ and $\|y\| > 1$. Suppose that for vectors $x_0 = \frac{x}{\|x\|}$ and $y_0 = \frac{y}{\|y\|}$ the inequality $\|x_0 + y_0\| < 2$ holds. Now we have

$$\begin{aligned}
\|x+y\| &= \|\|\|x\|x_0 + \|y\|y_0\| \\
&= \|(x_0 + y_0) + (\|x\| - 1)x_0 + (\|y\| - 1)y_0\| \\
&\le \|x_0 + y_0\| + (\|x\| - 1)\|x_0\| + (\|y\| - 1)\|y_0\| \\
&< 2 + \|x\| - 1 + \|y\| - 1 = \|x\| + \|y\|.
\end{aligned}$$

This contradicts the initial assumption $\|x+y\| = \|x\| + \|y\|$. As it is true that $\|x_0 + y_0\| \le 2$, we conclude that it must be $\|x_0 + y_0\| = 2$. □

Lemma 3.3. *Let X be a normed linear space and $x, y \in S_X$. It holds $[x, y] \subset S_X$ if and only if $\|x+y\| = 2$.*

Proof. Let $x, y \in S_X$ be such that $\|x+y\| = 2$. For an arbitrary $\lambda \in [0,1]$ it holds

$$\|(1-\lambda)x + \lambda y\| \le 1.$$

Suppose that there exists $\lambda_0 \in (0,1)$ such that $\|(1-\lambda_0)x + \lambda_0 y\| < 1$. Then we have

$$\begin{aligned}
2 = \|x+y\| &= \|(1-\lambda_0)x + \lambda_0 y + \lambda_0 x + (1-\lambda_0)y\| \\
&\le \|(1-\lambda_0)x + \lambda_0 y\| + \|\lambda_0 x + (1-\lambda_0)y\| \\
&< 1 + \lambda_0 + (1-\lambda_0) = 2,
\end{aligned}$$

which is a contradiction. Therefore, for all $\lambda \in [0,1]$ it holds $(1-\lambda)x + \lambda y \in S_X$, that is $[x, y] \subset S_X$.

Conversely, assume that $[x, y] \subset S_X$ $(x \neq y)$. Then $\frac{x+y}{2} \in S_X$, that is $\|x+y\| = 2$. □

Lemma 3.4. *Let X be a normed linear space and $x, y \in X$ be such that the equality $\|x+y\| = \|x\| + \|y\|$ holds. Then, for arbitrary $\alpha, \beta \in \mathbb{R}^+$ it holds*

$$\|\alpha x + \beta y\| = \alpha\|x\| + \beta\|y\|.$$

Proof. Let $x, y \in X$ be such that

$$\|x+y\| = \|x\| + \|y\|. \tag{5}$$

For arbitrary $\alpha, \beta \in \mathbb{R}^+$ then it holds

$$\|\alpha x + \beta y\| \leq \alpha\|x\| + \beta\|y\|. \tag{6}$$

If the vectors x and y are collinear, that is, $x = \lambda y$, $\lambda \in \mathbb{R}^+$, then we have

$$\begin{aligned} \|\alpha x + \beta y\| &= \|\alpha\lambda y + \beta y\| \\ &= |\alpha\lambda + \beta| \, \|y\| = \alpha\|\lambda y\| + \beta\|y\| \\ &= \alpha\|x\| + \beta\|y\|. \end{aligned}$$

Now, suppose that x and y are not collinear. Based on Proposition 3.1, this means that X is not an (SC) space. Let $\alpha, \beta \in \mathbb{R}^+$ be such that

$$\|\alpha x + \beta y\| < \alpha\|x\| + \beta\|y\|. \tag{7}$$

Let $\lambda_1, \lambda_2 \in \mathbb{R}^+$ be such that $x = \lambda_1 x_0$ and $y = \lambda_2 y_0$, where $x_0, y_0 \in S_X$. Based on the Lemma 3.3 and Lemma 3.2, we have $[x_0, y_0] \subset S_X$. The inequality (7) becomes

$$\|\alpha x + \beta y\| < \alpha\lambda_1\|x_0\| + \beta\lambda_2\|y_0\| = \alpha\lambda_1 + \beta\lambda_2.$$

Dividing this inequality by $\alpha\lambda_1 + \beta\lambda_2 \neq 0$ we get

$$\left\| \frac{\alpha\lambda_1}{\alpha\lambda_1 + \beta\lambda_2} x_0 + \frac{\beta\lambda_2}{\alpha\lambda_1 + \beta\lambda_2} y_0 \right\| < 1. \tag{8}$$

Because the vector $\overline{x} = \frac{\alpha\lambda_1}{\alpha\lambda_1+\beta\lambda_2} x_0 + \frac{\beta\lambda_2}{\alpha\lambda_1+\beta\lambda_2} y_0$ is a convex combination of vectors x_0 and y_0, this means $\overline{x} \in S_X$, which contradicts (8). So, using (6), we get

$$\|\alpha x + \beta y\| = \alpha\|x\| + \beta\|y\| \, . \qquad \square$$

Theorem 3.5. *Let X be a uniformly convex space and let $A \subset X$ be a convex, closed set with $d(0, A) > 0$. For $x^* \in A$ such that $d(0, A) = \|x^*\|$, we denote*

$$A_{x^*} = \{z \in A \mid z = \lambda x^* \text{ for some } 1 \leq \lambda < +\infty\} \, .$$

Then for all $x \in A \setminus A_{x^*}$ we have

$$d(0, A + x) < d(0, A) + \|x\| \, .$$

Proof. For arbitrary $x \in A$, using Lemma 2.4, we have

$$d(0, A + x) \leq d(0, A) + \|x\|$$

and if $x \in A_{x^*}$, then

$$d(0, A + x) = d(0, A) + \|x\| \, .$$

So, let $\overline{x} \in A \setminus A_{x^*}$ be arbitrary and let us suppose that

$$d(0, A + \overline{x}) = d(0, A) + \|\overline{x}\| \, .$$

Since $A+\overline{x} \subset X$ is convex and closed, because of uniform convexity of the space, there exists $y^* = x' + \overline{x} \in A + \overline{x}$ such that

$$\|y^*\| = d(0, A+\overline{x}) = \|x' + \overline{x}\| = d(0,A) + \|\overline{x}\| = \|x^*\| + \|\overline{x}\|\,. \tag{9}$$

Since every (UC) space is also an (SC) space, and since vectors x^* and $\overline{x}$ are not collinear it holds

$$\|x^* + \overline{x}\| < \|x^*\| + \|\overline{x}\|\,. \tag{10}$$

Using (9) and (10), we conclude that

$$\|x^* + \overline{x}\| < \|x' + \overline{x}\| = \|y^*\| = d(0, A+\overline{x})\,.$$

Since $x^* + \overline{x} \in A + \overline{x}$, this is an obvious contradiction, so we have

$$d(0, A+\overline{x}) < d(0,A) + \|\overline{x}\|\,. \qquad \square$$

For simplicity we use the following notation. We say that a normed space X has the translation property, X is a (TP) space for short, if every convex, closed set A with positive distance from 0, such that there exists $x^* \in A$ with $d(0,A) = \|x^*\|$, satisfies the condition $d(0, A+x) < d(0,A) + \|x\|$ for all $x \in A \setminus A_{x^*}$. Now, Theorem 3.5 tells us that every (UC) space is a (TP) space.

Consider the space l_1 with the norm

$$\|x\| = \|x\|_{l_1} + \|x\|_{l_2} = \sum_{n=1}^{\infty} |x_n| + \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{\frac{1}{2}}\,.$$

It is clear that for arbitrary $x \in l_1$ we have

$$\|x\|_{l_1} \leq \|x\| \leq 2\|x\|_{l_1}\,,$$

so these two norms are equivalent. The standard space l_1 is not reflexive, therefore l_1 is nonreflexive after renorming with the equivalent norm $\|\cdot\|$. Consequently, the renormed space l_1 is not a (UC) space, since uniform convexity implies reflexivity ([1]).

We will show that the renormed space l_1 is a (TP) space. Let $A \subset l_1$ be a convex, closed set for which exists $x^* = (\xi_n)_{n\in\mathbb{N}} \in A$ such that $d(0,A) = \|x^*\| > 0$. Suppose there exists $z = (z_n)_{n\in\mathbb{N}} \in A \setminus A_{x^*}$ with $d(0, A+z) = d(0,A) + \|z\|$. Then,

$$d(0,A+z) = d(0,A) + \|z\| = \|x^*\| + \|z\| \geq \|x^* + z\| \geq d(0, A+z)\,.$$

We conclude that $\|x^* + z\| = \|x^*\| + \|z\|$. Hence,

$$\sum_{n=1}^{\infty} |\xi_n + z_n| + \left(\sum_{n=1}^{\infty} |\xi_n + z_n|^2\right)^{\frac{1}{2}} = \sum_{n=1}^{\infty} |\xi_n| + \sum_{n=1}^{\infty} |z_n| + \left(\sum_{n=1}^{\infty} |\xi_n|^2\right)^{\frac{1}{2}} + \left(\sum_{n=1}^{\infty} |z_n|^2\right)^{\frac{1}{2}}\,.$$

Equality in the general Minkowski inequality holds if and only if $\xi_i \cdot z_i \geq 0$ $(i \in \mathbb{N})$, for $p = 1$, that is $|z_i|^p = c|\xi_i|^p$ $(i \in \mathbb{N})$, for $p > 1$. In our case this means that for all $n \in \mathbb{N}$ we have $z_n = c\xi_n$ for some $c > 0$, i.e., $z = cx^*$. This is a contradiction with the choice of the element $z \in A \setminus A_{x^*}$. Thus, for an arbitrary $z \in A \setminus A_{x^*}$,

$$d(0, A + z) < d(0, A) + \|z\|$$

holds and because of arbitrariness of the set A, this establishes the translation property of the renormed space l_1. This proves that there exists a (TP) space that is not a (UC) space.

Theorem 3.6. *Let X be a normed linear space and let $A \subset X$ be a convex, closed set with $d(0, A) > 0$ such that there exists $x^* \in A$ with $\|x^*\| = d(0, A)$. The inequality*

$$d(0, A + x) < d(0, A) + \|x\|\,,$$

holds for all $x \in A \setminus A_{x^*}$ if and only if the space X is strictly convex.

Proof. Suppose that the space X is not a (TP) space. This means that there exists a closed and convex set $A \subset X$ for which there exists $x^* \in A$ such that $d(0, A) = \|x^*\| > 0$ and there exists $\overline{x} \in A\setminus A_{x^*}$ such that

$$d(0, A + \overline{x}) = d(0, A) + \|\overline{x}\|\,.$$

Therefore,

$$d(0, A + \overline{x}) = \|x^*\| + \|\overline{x}\| \geq \|x^* + \overline{x}\| \geq d(0, A + \overline{x})\,.$$

We conclude that

$$\|x^*\| + \|\overline{x}\| = \|x^* + \overline{x}\|\,.$$

Since x^* and $\overline{x}$ are not collinear, we conclude that the space X is not strictly convex. Using contraposition, we get that if X is (SC), then X is (TP).

Suppose now that space X is not strictly convex. This means that there exist $x, y \in S_X$, $x \neq y$ and $[x, y] \subset S_X$ or equivalently

$$(\forall \lambda \in [0, 1])\ \lambda x + (1 - \lambda)y \in S_X\,.$$

Consider the set $A = \overline{\operatorname{conv}}\left\{x, y, \frac{x+y}{4}\right\}$. It is obvious that A is a convex and closed set. Suppose now that there exists $a \in A$ with $\|a\| < \frac{1}{2}$. This means that there are $\mu_1, \mu_2, \mu_3 \geq 0$ such that

$$a = \mu_1 x + \mu_2 y + \mu_3 x^*\,,\ \mu_1 + \mu_2 + \mu_3 = 1\,,$$

where $x^* = \frac{x+y}{4}$. Then we have

$$\|a\| = \left\|\mu_1 x + \mu_2 y + \mu_3 \frac{x + y}{4}\right\| = \left\|\left(\mu_1 + \frac{\mu_3}{4}\right) x + \left(\mu_2 + \frac{\mu_3}{4}\right) y\right\|\,.$$

Based on Lemma 3.4, we now have

$$\|a\| = \mu_1 + \mu_2 + \frac{\mu_3}{2} = 1 - \frac{\mu_3}{2} < \frac{1}{2},$$

which means that $\mu_3 > 1$. This is of course impossible due to the conditions on the coefficients in the decomposition of the vector a. So for every $a \in A$ we have $\|a\| \geq \frac{1}{2}$. Because $x^* \in A$ and

$$\|x^*\| = \left\| \frac{1}{2}\left(\frac{x+y}{2}\right) \right\| = \frac{1}{2},$$

since $\frac{x+y}{2} \in S_X$, we conclude that $d(0,A) = \frac{1}{2}$.

Since x and x^* are not collinear, we get $x \in A \setminus A_{x^*}$. Let us make the translation of the set A by the vector x. We have

$$\left\|\frac{x}{2}\right\| = \frac{1}{2} = \|x^*\| \quad \text{and} \quad \left\|\frac{x}{2} + x\right\| = \frac{3}{2}\,, \ \|x + x^*\| \leq \frac{3}{2}.$$

From Lemma 3.4 we have

$$\|x + x^*\| = \left\|\frac{5}{4}x + \frac{1}{4}y\right\| = \frac{5}{4} + \frac{1}{4} = \frac{3}{2}. \tag{11}$$

Suppose there exists $a \in A$ such that $\|a + x\| < \|x^* + x\|$. Then $a = \mu_1 x + \mu_2 y + \mu_3 x^*$, where $\mu_1 + \mu_2 + \mu_3 = 1$ and $\mu_1, \mu_2, \mu_3 \geq 0$. Now again based on Lemma 3.4, we have

$$\begin{aligned} \|a + x\| &= \left\| \left(\mu_1 + \frac{\mu_3}{4} + 1\right) x + \left(\mu_2 + \frac{\mu_3}{4}\right) y \right\| \\ &= \mu_1 + \mu_2 + \frac{\mu_3}{2} + 1 = 2 - \frac{\mu_3}{2}. \end{aligned}$$

Due to the assumption $\|a + x\| < \frac{3}{2}$ we now have $2 - \dfrac{\mu_3}{2} < \dfrac{3}{2}$, which is equivalent to $\mu_3 > 1$. This is a contradiction with the choice of the coefficient μ_3.

Therefore, for any $a \in A$ we have $\|a + x\| \geq \|x^* + x\| = \frac{3}{2}$. Because $x^* \in A$, we conclude that

$$d(0, A + x) = \inf_{a \in A} \|a + x\| = \|x^* + x\|. \tag{12}$$

Using (11) and (12), we conclude that

$$d(0, A + x) = d(0, A) + \|x\|\,.$$

Therefore, if the space is not an (SC) space, then it is possible to construct a convex, closed set A such that $d(0,A) = \|x^*\| > 0$ for some $x^* \in A$, so that there exists $x \in A \setminus A_{x^*}$ for which $d(0, A + x) = d(0, A) + \|x\|$. This means that the space is not a (TP) space. Using contraposition, we get the result. $\square$

Theorem 3.6 tells us that the translation property is equivalent to strict convexity, which gives a new characterization of strict convexity.

References

[1] Ayerbe Toledano, J. M., Domínguez Benavides, T., López Acedo, G., *Measures of Noncompactness in Metric Fixed Point Theory*, Birkhäuser Verlag, Basel, 1997.

[2] Cobzas, S., *Geometric properties of Banach spaces and the existence of nearest and farthest points*, Abstr. Appl. Anal. **2005**(3) (2005), 259–285.

[3] Istrăţescu, V. I., *Strict Convexity and Complex Strict Convexity: Theory and Applications*, Marcel Dekker, Inc., New York, 1984.

Nermin Okičić
Department of Mathematics
Faculty of Natural Sciences
and Mathematics
University of Tuzla
Univerzitetska 4, 75000 Tuzla
Bosnia and Herzegovina
e-mail: `nermin.okicic@untz.ba`

Amra Rekić-Vuković
Department of Mathematics
Faculty of Natural Sciences
and Mathematics
University of Tuzla
Univerzitetska 4, 75000 Tuzla
Bosnia and Herzegovina
e-mail: `amra.rekic@untz.ba`

Vedad Pašić
Department of Mathematics
Faculty of Natural Sciences
and Mathematics
University of Tuzla
Univerzitetska 4, 75000 Tuzla
Bosnia and Herzegovina
e-mail: `vedad.pasic@untz.ba`

Received February 23, 2022