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Generalized Kaplan classes and their applications

ABSTRACT. Ali and Vasudevarao considered the integral operator $I_{r,s}(z) := \int_0^z (f'(t))^r (g'(t))^s \mathrm{d}t$ and determined all values of r and s for which the operator $(f, g) \mapsto I_{r,s}$ maps a specified subclass of Hornich space into another specified subclass of Hornich space. Thus, as it was stated by Kumar and Sahoo, Ali and Vasudevarao studied the range of r and s that preserves properties of these specified classes. Based on the Kaplan classes, we introduce the product classes $K_{a,b}$ for arbitrary finite sequences a and b and consider operations similar to Hornich operations. To this end we improve Sheil-Small's factorization theorem. Moreover, using elaborated techniques, we simplify proofs and solve the generalized problems considered by Causey and Reade, Goodman, Kim and Merkes.

1. Introduction. In this section, we define several symbols and prove some auxiliary lemmas.

1.1. Basic definitions. We consider the following subclasses of the class of all analytic functions in the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$:

- $\mathcal{A}$ is the class of all functions f normalized by $f(0) = f'(0) - 1 = 0$,
- $\mathcal{H}$ is the subclass of $\mathcal{A}$ of all functions f that are locally univalent, i.e. $f' \neq 0$ in $\mathbb{D}$,
- $\mathcal{S}$ is the class of all univalent functions belonging to $\mathcal{A}$,
- $\mathcal{K}$ is the class of functions in $\mathcal{S}$ that map $\mathbb{D}$ onto a convex set,

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- $\mathcal{S}^*$ is the class of functions in $\mathcal{S}$ that map $\mathbb{D}$ onto a starlike domain with respect to 0,
- $\mathcal{C}$ is the class of functions in $\mathcal{S}$ that are close-to-convex,
- $\mathcal{H}_d$ is the class of all analytic functions f normalized by $f(0) = 1$ and such that $f \neq 0$ in $\mathbb{D}$.

The Kaplan classes were one of the means used as a universal tool for establishing many important subclasses of $\mathcal{S}$ (see [18, p. 47]). Since then, many authors have studied properties of Kaplan classes, in particular:

- Sheil-Small established a theorem on factorization of the Kaplan classes (see [20, p. 246]),
- Kim et al. posed many problems regarding the properties of integral operators (see [11]), while Lamprecht in collaboration with Ruscheweyh (see [14]), using Kaplan classes, solved one of the open problems stated by Kim,
- Ruscheweyh and Sumyk studied classes $K(\alpha, \beta)$, among others, in the context of their relations with classes $T(\alpha, \beta)$ (see [19]).

Based on the results of Ruscheweyh (see [18]), Sheil-Small (see [20]), Lamprecht (see [14]), Ruscheweyh and Sumyk (see [19]), we extend the properties of the Kaplan classes, among others, in relation to the factorization problem of these classes (see Section 2). Then we introduce the generalized Kaplan classes $K_{a,b}$ for arbitrary finite sequences a and b. The generalized Kaplan classes $K_{a,b}$ are based on Kaplan classes and operations similar to Hornich operations. We show that the generalized Kaplan classes $K_{a,b}$ preserve membership to a given Kaplan class $K(\alpha, \beta)$. It corresponds to the idea proposed by Ali and Vasudevarao [1] (see also [13]).

In Section 3 we apply these results to simplify proofs and solve the generalized problems of univalence of integral operators (see also [2, 3, 6, 12, 15, 16]).

1.2. Definition of the Kaplan classes. For $\alpha, \beta \geq 0$ the Kaplan class $K(\alpha, \beta)$ is the set of all functions $f \in \mathcal{H}_d$ satisfying the condition

$$-\alpha\pi - \frac{1}{2}(\alpha - \beta)(\theta_1 - \theta_2) \leq \arg f(r\mathrm{e}^{\mathrm{i}\theta_2}) - \arg f(r\mathrm{e}^{\mathrm{i}\theta_1}) \tag{1.1}$$

for $0 < r < 1$ and $\theta_1 < \theta_2 < \theta_1 + 2\pi$ (see [18, pp. 32–33]).

Let us recall [18, p. 46] that

- $f \in \mathcal{K}$ if and only if $f' \in K(0, 2)$,
- $f \in \mathcal{C}$ if and only if $f' \in K(1, 3)$,
- $f \in \mathcal{S}^*$ if and only if $f/\operatorname{Id} \in K(0, 2)$,
  where $\mathbb{D} \ni z \mapsto \operatorname{Id}(z) := z$.

1.3. Alternative definition of the Kaplan classes. Let us notice that the condition (1.1) can be written in an equivalent way (see [20, p. 245]).

Lemma 1.1. *For every $\alpha,\beta \geq 0$ and function $f \in \mathcal{H}_d$ the following inequalities are equivalent:*

$$-\alpha\pi - \frac{\alpha-\beta}{2}(\theta_1-\theta_2) \leq \arg f(re^{i\theta_2}) - \arg f(re^{i\theta_1})\,; \tag{1.2}$$

$$\arg f(re^{i\theta_2}) - \arg f(re^{i\theta_1}) \leq \beta\pi - \frac{\alpha-\beta}{2}(\theta_1-\theta_2)\,; \tag{1.3}$$

$$\begin{aligned}-\alpha\pi - \frac{\alpha-\beta}{2}(\theta_1-\theta_2) &\leq \arg f(re^{i\theta_2}) - \arg f(re^{i\theta_1})\\ &\leq \beta\pi - \frac{\alpha-\beta}{2}(\theta_1-\theta_2)\end{aligned} \tag{1.4}$$

for $0<r<1$ and $\theta_1<\theta_2<\theta_1+2\pi$.

Proof. It is enough to prove that inequalities (1.2) and (1.3) are equivalent. Fix $r \in (0;1)$ and $\theta_1<\theta_2<\theta_1+2\pi$. Using (1.2) with θ_1 and θ_2 replaced by θ_2 and $\theta_1+2\pi$, respectively, we get

$$-\alpha\pi - \frac{1}{2}(\alpha-\beta)(\theta_2-\theta_1-2\pi) \leq \arg f(re^{i\theta_1+2\pi}) - \arg f(re^{i\theta_2})\,.$$

Hence

$$-\alpha\pi + \pi(\alpha-\beta) - \frac{1}{2}(\alpha-\beta)(\theta_2-\theta_1) \leq \arg f(re^{i\theta_1}) - \arg f(re^{i\theta_2})$$

and consequently

$$\beta\pi - \frac{1}{2}(\alpha-\beta)(\theta_1-\theta_2) \geq \arg f(re^{i\theta_2}) - \arg f(re^{i\theta_1})$$

for all $r \in (0;1)$ and $\theta_1<\theta_2<\theta_1+2\pi$, which ends the proof. □

Inequalities (1.2), (1.3), (1.4) give us a better view on relations between Kaplan classes and possible operations, which can be defined using these classes. The properties listed below were described in [20], but in this work we complement and extend them. First we will need the following lemma.

Lemma 1.2. *For all $\alpha_1,\alpha_2,\beta_1,\beta_2 \geq 0$ and $t \in \mathbb{R}\backslash\{0\}$ the following conditions hold:*

$$f\in K(\alpha_1,\beta_1) \text{ and } g\in K(\alpha_2,\beta_2) \Rightarrow fg\in K(\alpha_1+\alpha_2,\beta_1+\beta_2)\,, \tag{1.5}$$

$$f\in K(\alpha_1,\beta_1) \Leftrightarrow f^t\in K\left(\frac{|t|+t}{2}\alpha_1+\frac{|t|-t}{2}\beta_1, \frac{|t|+t}{2}\beta_1+\frac{|t|-t}{2}\alpha_1\right), \tag{1.6}$$

$$f\in K(\alpha_1,\beta_1) \Rightarrow f^0\in K(0,0)\,. \tag{1.7}$$

Proof. The implication (1.5) was proved in [14]. Now we prove the equivalence (1.6). Fix $t>0$. Multiplying the inequality (1.1) by t, we get

$$-t\alpha_1\pi - t\frac{1}{2}(\alpha_1-\beta_1)(\theta_1-\theta_2) \leq t\arg f(re^{i\theta_2}) - t\arg f(re^{i\theta_1})$$

and as a consequence

$$-t\alpha_1\pi - t\frac{1}{2}(\alpha_1-\beta_1)(\theta_1-\theta_2) \le \arg f^t(r\mathrm{e}^{\mathrm{i}\theta_2}) - \arg f^t(r\mathrm{e}^{\mathrm{i}\theta_1})\,.$$

Hence by (1.2) we have

$$f \in K(\alpha_1,\beta_1) \iff f^t \in K(t\alpha_1, t\beta_1)\,. \tag{1.8}$$

Fix $t<0$. Multiplying the inequality (1.1) by t, we obtain

$$-t\alpha_1\pi - t\frac{1}{2}(\alpha_1-\beta_1)(\theta_1-\theta_2) \ge t\arg f(r\mathrm{e}^{\mathrm{i}\theta_2}) - t\arg f(r\mathrm{e}^{\mathrm{i}\theta_1})$$

and so

$$|t|\alpha_1\pi - \frac{1}{2}(|t|\beta_1-|t|\alpha_1)(\theta_1-\theta_2) \ge \arg f^t(r\mathrm{e}^{\mathrm{i}\theta_2}) - \arg f^t(r\mathrm{e}^{\mathrm{i}\theta_1})\,.$$

Hence by (1.3) we get

$$f \in K(\alpha_1,\beta_1) \iff f^t \in K(|t|\beta_1, |t|\alpha_1)\,. \tag{1.9}$$

From equivalences (1.8) and (1.9) we obtain the condition (1.6) for $t\in\mathbb{R}\backslash\{0\}$ and $\alpha_1,\beta_1\ge 0$.

Assume that $f\in K(\alpha_1,\beta_1)$. Then by definition of the Kaplan classes, $f\neq 0$ in $\mathbb{D}$. Therefore $f^0\equiv 1$ satisfies the condition (1.4) with $\alpha=\beta=0$, which leads to $f^0\in K(0,0)$. □

1.4. Inclusions for Kaplan classes. Now we define and classify functions corresponding to the appropriate Kaplan classes, which is necessary to study univalence of integral operators.

Lemma 1.3. *The function* $\mathbb{D}\ni z\mapsto \delta(z) := 1-z$ *satisfies the following properties:*

$$\delta \in K(1,0) \tag{1.10}$$

and for $0\le\alpha<1$, $0\le\beta$,

$$\delta \notin K(\alpha,\beta)\,. \tag{1.11}$$

Proof. Let $\mathbb{D}\ni z\mapsto f(z) := z/(1-z)$. Since $f\in\mathcal{K}$, it follows that $f'\in K(0,2)$. Since $\delta=(f')^{-1/2}$, we see by the equivalence (1.6) that $\delta\in K(1,0)$.

Now we prove (1.11). Using (1.3) with r, θ_1 and θ_2 replaced by $\mathbb{N}\ni n\mapsto r(n) := 1-1/(2n^2)$, $\mathbb{N}\ni n\mapsto\theta_1(n) := 1/n$ and $\mathbb{N}\ni n\mapsto\theta_2(n) := 2\pi-1/n$, respectively, from the equality

$$\overline{\mathrm{e}^{\mathrm{i}\theta_2(n)}} = \mathrm{e}^{\mathrm{i}\theta_1(n)}$$

we deduce that

$$\arg\left(\delta\left(r(n)\mathrm{e}^{\mathrm{i}\theta_2(n)}\right)\right) - \arg\left(\delta\left(r(n)\mathrm{e}^{\mathrm{i}\theta_1(n)}\right)\right) = -2\arg\left(1-\left(1-\frac{1}{2n^2}\right)\mathrm{e}^{\mathrm{i}/n}\right).$$

Since

$$\operatorname{Re}\left(1-\left(1-\frac{1}{2n^2}\right)\mathrm{e}^{\mathrm{i}/n}\right)=1-\left(1-\frac{1}{2n^2}\right)\cos\left(\frac{1}{n}\right)>0$$

and

$$\operatorname{Im}\left(1-\left(1-\frac{1}{2n^2}\right)\mathrm{e}^{\mathrm{i}/n}\right)=-\left(1-\frac{1}{2n^2}\right)\sin\left(\frac{1}{n}\right),$$

we obtain

$$-2\arg\left(1-\left(1-\frac{1}{2n^2}\right)\mathrm{e}^{\mathrm{i}/n}\right)=2\arctan\left(\frac{\left(1-\frac{1}{2n^2}\right)\sin\left(\frac{1}{n}\right)}{1-\left(1-\frac{1}{2n^2}\right)\cos\left(\frac{1}{n}\right)}\right).$$

Since $\operatorname{Re}(1-z)>0$ for $z\in\mathbb{D}$ and

$$\lim_{n\to+\infty} 2\arctan\left(\frac{\left(1-\frac{1}{2n^2}\right)\sin\left(\frac{1}{n}\right)}{1-\left(1-\frac{1}{2n^2}\right)\cos\left(\frac{1}{n}\right)}\right)=\pi\,, \tag{1.12}$$

we get

$$\pi=\sup\left\{\delta(r\mathrm{e}^{\mathrm{i}\theta_2})-\delta(r\mathrm{e}^{\mathrm{i}\theta_1}):\theta_1<\theta_2<\theta_1+2\pi\right\}.$$

Now consider the right side of the inequality (1.3) with $f:=\delta$. Fix $\alpha,\beta\geq 0$. Then

$$\beta\pi-\frac{1}{2}(\alpha-\beta)(\theta_1(n)-\theta_2(n))=\alpha\left(\pi-\frac{1}{n}\right)+\frac{\beta}{n}$$

and consequently

$$\lim_{n\to+\infty}\alpha\left(\pi-\frac{1}{n}\right)+\frac{\beta}{n}=\alpha\pi\,. \tag{1.13}$$

From conditions (1.12) and (1.13) we deduce that the inequality (1.3) does not hold for $f:=\delta$, $0\leq\alpha<1$ and $\beta\geq 0$. □

Using the function δ, we can prove some relations between Kaplan classes.

Lemma 1.4. *For all $\alpha_1,\alpha_2,\beta_1,\beta_2\geq 0$ the following equivalences hold:*

$$\alpha_1\leq\alpha_2 \iff K(\alpha_1,\beta_1)\subset K(\alpha_2,\beta_1)\,, \tag{1.14}$$

$$\beta_1\leq\beta_2 \iff K(\alpha_1,\beta_1)\subset K(\alpha_1,\beta_2)\,. \tag{1.15}$$

Proof. Let $0\leq\alpha_1\leq\alpha_2$ and $\beta_1\geq 0$. Since $\theta_1<\theta_2$, from (1.3) we see that

$$\arg f(r\mathrm{e}^{\mathrm{i}\theta_2})-\arg f(r\mathrm{e}^{\mathrm{i}\theta_1})\leq\beta_1\pi-\frac{\alpha_1-\beta_1}{2}(\theta_1-\theta_2)\leq\beta_1\pi-\frac{\alpha_2-\beta_1}{2}(\theta_1-\theta_2)\,,$$

from which

$$\alpha_1\leq\alpha_2\Rightarrow K(\alpha_1,\beta_1)\subset K(\alpha_2,\beta_1)\,. \tag{1.16}$$

Let $0\leq\beta_1\leq\beta_2$ and $\alpha_1\geq 0$. Since $\theta_1<\theta_2$, from (1.2) we deduce that

$$\begin{aligned}-\alpha_1\pi-\frac{\alpha_1-\beta_2}{2}(\theta_1-\theta_2)&\leq-\alpha_1\pi-\frac{\alpha_1-\beta_1}{2}(\theta_1-\theta_2)\\&\leq\arg f(r\mathrm{e}^{\mathrm{i}\theta_2})-\arg f(r\mathrm{e}^{\mathrm{i}\theta_1})\end{aligned}$$

and so

$$\beta_1 \le \beta_2 \Rightarrow K(\alpha_1,\beta_1) \subset K(\alpha_1,\beta_2)\,. \tag{1.17}$$

Suppose that $0 \le \alpha_2 < \alpha_1$ and $\beta_1, \beta_2 \ge 0$. Then by (1.6), (1.11) and (1.17) we get $\delta^{\alpha_1} \in K(\alpha_1,\beta_1)$ and $\delta^{\alpha_1} \notin K(\alpha_2,\beta_1)$. This ends the proof of the equivalence (1.14) in the direction ($\Leftarrow$).

Suppose that $0 \le \beta_2 < \beta_1$ and $\alpha_1, \alpha_2 \ge 0$. Then by equivalence (1.6), (1.11) and (1.16), $1/\delta^{\beta_1} \in K(\alpha_1,\beta_1)$ and $1/\delta^{\beta_1} \notin K(\alpha_1,\beta_2)$. This ends the proof of equivalence (1.15) in the direction ($\Leftarrow$). □

As a matter of fact, Sheil-Small proved in [20, p. 245] the following implication

$$\alpha_1 \le \alpha_2 \text{ and } \beta_1 \le \beta_2 \Rightarrow K(\alpha_1,\beta_1) \subset K(\alpha_2,\beta_2)$$

for $\alpha_1, \alpha_2, \beta_1, \beta_2 \ge 0$. From Lemma 1.4 we improve this result as follows.

Corollary 1.5. *For all $\alpha_1, \alpha_2, \beta_1, \beta_2 \ge 0$ the following equivalence holds*

$$\alpha_1 \le \alpha_2, \beta_1 \le \beta_2 \iff K(\alpha_1,\beta_1) \subset K(\alpha_2,\beta_2)\,. \tag{1.18}$$

2. Main theorems. In this section we present several theorems about preserving Kaplan classes.

2.1. Generalized Kaplan classes. Let $\mathbb{R}_0^+ := [0;+\infty)$ and $\mathbb{N}_n := \mathbb{N} \cap [1;n]$ for every $n \in \mathbb{N}$.

Definition 2.1. For $n \in \mathbb{N}$ and $a, b : \mathbb{N}_n \to \mathbb{R}_0^+$ the class

$$K_{a,b} = K_{(a_1,\dots,a_n),(b_1,\dots,b_n)} := \left\{ \prod_{k=1}^{n} f_k : f_k \in K(a_k,b_k) \text{ for } k \in \mathbb{N}_n \right\} \tag{2.1}$$

is said to be the generalized Kaplan class for sequences a and b.

Of course for every $\alpha, \beta \ge 0$ the equality $K(\alpha,\beta) = K_{(\alpha),(\beta)}$ holds, so the Kaplan classes are the special case of the generalized Kaplan classes. Now we define two inner operations on

$$\mathcal{D}_{\mathcal{K}} := \bigcup_{n\in\mathbb{N}} \bigcup_{a,b:\mathbb{N}_n\to\mathbb{R}_0^+} K_{a,b}\,,$$

which are not inner for Kaplan classes. For $m, n \in \mathbb{N}$, $a, b : \mathbb{N}_m \to \mathbb{R}_0^+$, $c, d : \mathbb{N}_n \to \mathbb{R}_0^+$ and $t \in \mathbb{R}$ we define

$$K_{a,b} \oplus K_{c,d} := \{f \cdot g : f \in K_{a,b}, g \in K_{c,d}\} \tag{2.2}$$

and

$$t \odot K_{a,b} := \{f^t : f \in K_{a,b}\}\,. \tag{2.3}$$

Operations given by (2.2) and (2.3) are similar to the Hornich operations in [8] (see also [14]). The difference is that in our paper $\oplus$ and $\odot$ are operations on classes of functions, not on functions. Later in this subsection we study the relationship between classes

$$K(\alpha_1,\beta_1)\oplus K(\alpha_2,\beta_2)\oplus\ldots\oplus K(\alpha_n,\beta_n)$$

and

$$K(\alpha_1+\alpha_2+\ldots+\alpha_n,\beta_1+\beta_2+\ldots+\beta_n)\,.$$

Lemma 2.2. *For all $\alpha_1,\alpha_2,\beta_1,\beta_2\ge 0$, if $\alpha_1\cdot\beta_2=\alpha_2\cdot\beta_1$, then the following equality holds*

$$K(\alpha_1,\beta_1)\oplus K(\alpha_2,\beta_2)=K(\alpha_1+\alpha_2,\beta_1+\beta_2)\,. \tag{2.4}$$

Proof. Fix $\alpha_1,\alpha_2,\beta_1,\beta_2\ge 0$ such that $\alpha_1\cdot\beta_2=\alpha_2\cdot\beta_1$. Assume that $f_1\in K(\alpha_1,\beta_1)$ and $f_2\in K(\alpha_2,\beta_2)$. Therefore, directly by (1.5) we get

$$f_1\cdot f_2\in K(\alpha_1+\alpha_2,\beta_1+\beta_2)\,,$$

from which

$$K(\alpha_1,\beta_1)\oplus K(\alpha_2,\beta_2)\subset K(\alpha_1+\alpha_2,\beta_1+\beta_2)\,.$$

Now assume that $g\in K(\alpha_1+\alpha_2,\beta_1+\beta_2)$. The case $\alpha_1=\alpha_2=\beta_1=\beta_2=0$ is trivial. If α_1, α_2, β_1 or β_2 is not equal to 0 then $\alpha_1+\alpha_2>0$ or $\beta_1+\beta_2>0$. Hence without loss of generality we can assume that $\alpha_1+\alpha_2>0$. Setting

$$f_1:=g^{\frac{\alpha_1}{\alpha_1+\alpha_2}}$$

and

$$f_2:=g^{\frac{\alpha_2}{\alpha_1+\alpha_2}}$$

we get

$$f_1\cdot f_2=g^{\frac{\alpha_1}{\alpha_1+\alpha_2}}\cdot g^{\frac{\alpha_2}{\alpha_1+\alpha_2}}=g\,. \tag{2.5}$$

Using the assumption $\alpha_1\cdot\beta_2=\alpha_2\cdot\beta_1$, we get

$$\frac{\alpha_1}{\alpha_1+\alpha_2}\cdot(\beta_1+\beta_2)=\frac{\alpha_1\beta_1+\alpha_1\beta_2}{\alpha_1+\alpha_2}=\frac{\alpha_1\beta_1+\alpha_2\beta_1}{\alpha_1+\alpha_2}=\beta_1 \tag{2.6}$$

and

$$\frac{\alpha_2}{\alpha_1+\alpha_2}\cdot(\beta_1+\beta_2)=\frac{\alpha_2\beta_1+\alpha_2\beta_2}{\alpha_1+\alpha_2}=\frac{\alpha_1\beta_2+\alpha_2\beta_2}{\alpha_1+\alpha_2}=\beta_2\,. \tag{2.7}$$

By (1.6), (1.8), (2.6) and (2.7), we get

$$f_1=g^{\frac{\alpha_1}{\alpha_1+\alpha_2}}\in K\left(\frac{\alpha_1}{\alpha_1+\alpha_2}\cdot(\alpha_1+\alpha_2),\frac{\alpha_1}{\alpha_1+\alpha_2}\cdot(\beta_1+\beta_2)\right)=K(\alpha_1,\beta_1)$$

and

$$f_2=g^{\frac{\alpha_2}{\alpha_1+\alpha_2}}\in K\left(\frac{\alpha_2}{\alpha_1+\alpha_2}\cdot(\alpha_1+\alpha_2),\frac{\alpha_2}{\alpha_1+\alpha_2}\cdot(\beta_1+\beta_2)\right)=K(\alpha_2,\beta_2)\,,$$

from which

$$K(\alpha_1+\alpha_2,\beta_1+\beta_2)\subset K(\alpha_1,\beta_1)\oplus K(\alpha_2,\beta_2)\,. \qquad \square$$

Lemma 2.2 leads to the following corollary.

Corollary 2.3. *For all* $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$ *the following equalities hold:*

$$K(\alpha_1, 0) \oplus K(\alpha_2, 0) = K(\alpha_1 + \alpha_2, 0)\,, \tag{2.8}$$

$$K(0, \beta_1) \oplus K(0, \beta_2) = K(0, \beta_1 + \beta_2)\,, \tag{2.9}$$

$$K(\alpha_1, \alpha_1) \oplus K(\alpha_2, \alpha_2) = K(\alpha_1 + \alpha_2, \alpha_1 + \alpha_2)\,. \tag{2.10}$$

Using the operations $\oplus$ and $\odot$, we can rephrase the factorization theorem from [20, p. 246] in the following way.

Theorem A (Sheil-Small, 2002). *For all* $\alpha, \beta \geq 0$

$$K(\alpha, \beta) = K(\min(\alpha, \beta), \min(\alpha, \beta)) \oplus ((\alpha - \beta) \odot K(1, 0))\,.$$

Now we prove the following theorem, which is a generalization of Theorem A.

Theorem 2.4. *For all* $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$, *if* $(\beta_1 - \alpha_1)(\beta_2 - \alpha_2) \geq 0$, *then the following equality holds*

$$K(\alpha_1, \beta_1) \oplus K(\alpha_2, \beta_2) = K(\alpha_1 + \alpha_2, \beta_1 + \beta_2)\,. \tag{2.11}$$

Proof. Given $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$ suppose that $(\beta_1 - \alpha_1)(\beta_2 - \alpha_2) > 0$. Hence the expressions $(\beta_1 - \alpha_1)$ and $(\beta_2 - \alpha_2)$ have the same sign. Setting $m_1 := \min(\alpha_1, \beta_1)$, $m_2 := \min(\alpha_2, \beta_2)$, $d_1 = \alpha_1 - \beta_1$ and $d_2 = \alpha_2 - \beta_2$, we deduce from Theorem A that the equality

$$\begin{aligned} &K(\alpha_1, \beta_1) \oplus K(\alpha_2, \beta_2) \\ &\qquad = K(m_1, m_1) \oplus (d_1 \odot K(1, 0)) \oplus K(m_2, m_2) \oplus (d_2 \odot K(1, 0)) \end{aligned}$$

holds for $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$. Since d_1 and d_2 have the same sign, we see by Corollary 2.3 that

$$\begin{aligned} K(m_1, m_1) &\oplus (d_1 \odot K(1, 0)) \oplus K(m_2, m_2) \oplus (d_2 \odot K(1, 0)) \\ &= K(m_1 + m_2, m_1 + m_2) \oplus ((d_1 + d_2) \odot K(1, 0))\,. \end{aligned}$$

Using Theorem A again, we get

$$K(m_1 + m_2, m_1 + m_2) \oplus ((d_1 + d_2) \odot K(1, 0)) = K(\alpha_1 + \alpha_2, \beta_1 + \beta_2)\,,$$

and so

$$K(\alpha_1, \beta_1) \oplus K(\alpha_2, \beta_2) = K(\alpha_1 + \alpha_2, \beta_1 + \beta_2)\,.$$

The case $(\beta_1 - \alpha_1)(\beta_2 - \alpha_2) = 0$ is obvious. □

The condition $(\beta_1 - \alpha_1)(\beta_2 - \alpha_2) \geq 0$ is sufficient for the equality (2.11). In the natural way the question arises: Does the equality (2.11) hold without this condition? If the answer to that question was positive, then Kaplan classes would be fully decomposable. In particular it means that any function with positive real part in $\mathbb{D}$ could be expressed as a square root of a quotient of derivatives of two convex functions from the class $\mathcal{K}$. Nevertheless, we will show in Theorem 2.7 that operation $\oplus$ in some sense preserves Kaplan classes, even for generalized Kaplan classes.

For $n \in \mathbb{N}$ and $a, b : \mathbb{N}_n \to \mathbb{R}_0^+$ we define

$$A_n := \sum_{k=1}^{n} a_k \,, \tag{2.12}$$

$$B_n := \sum_{k=1}^{n} b_k \,. \tag{2.13}$$

From the implication (1.5) we directly obtain the following lemma.

Lemma 2.5. *For $n \in \mathbb{N}$ and $a, b : \mathbb{N}_n \to \mathbb{R}_0^+$ the following inclusion holds*

$$K_{a,b} \subset K(A_n, B_n) \,. \tag{2.14}$$

The inclusion (2.14) is sharp in some sense, as stated in the following lemma.

Lemma 2.6. *For $n \in \mathbb{N}$ and $a, b : \mathbb{N}_n \to \mathbb{R}_0^+$ the following inclusions hold:*

$$K(A_n, 0) \subset K_{a,b} \,, \tag{2.15}$$

$$K(0, B_n) \subset K_{a,b} \,. \tag{2.16}$$

Proof. Suppose that $n \in \mathbb{N}$, $a, b : \mathbb{N}_n \to \mathbb{R}_0^+$ and $f \in K(A_n, 0)$. Set $f_k := f^{a_k/A_n}$ for every $k \in \mathbb{N}_n$. By (1.2), for all $0 < r < 1$ and $\theta_1 < \theta_2 < \theta_1 + 2\pi$, we obtain

$$-A_n\pi - \frac{1}{2}A_n(\theta_1 - \theta_2) \le \arg f\left(re^{i\theta_2}\right) - \arg f\left(re^{i\theta_1}\right),$$

from which

$$-a_k\pi - \frac{1}{2}a_k(\theta_1 - \theta_2) \le \arg f^{\frac{a_k}{A_n}}\left(re^{i\theta_2}\right) - \arg f^{\frac{a_k}{A_n}}\left(re^{i\theta_1}\right).$$

Hence for $k \in \mathbb{N}_n$, we obtain

$$f^{\frac{a_k}{A_n}} \in K(a_k, 0) \,.$$

Therefore by (1.15) we get $f^{a_k/A_n} \in K(a_k, b_k)$. Since

$$f = \prod_{k=1}^{n} f^{\frac{a_k}{A_n}} \,,$$

$f \in K_{a,b}$, which leads to (2.15). The inclusion (2.16) can be proved analogously. □

In the following theorem we show that operations $\oplus$ on classes $K(a_k, b_k)$ hidden in $K_{a,b}$ preserve the class $K(A_n, B_n)$.

Theorem 2.7. *For $n \in \mathbb{N}$, $a, b : \mathbb{N}_n \to \mathbb{R}_0^+$ and $\alpha, \beta \geq 0$ the following equivalence hold*

$$K_{a,b} \subset K(\alpha, \beta) \iff A_n \leq \alpha \text{ and } B_n \leq \beta \,. \tag{2.17}$$

Proof. Suppose that $n \in \mathbb{N}$, $a, b : \mathbb{N}_n \to \mathbb{R}_0^+$ and $\alpha, \beta \geq 0$. The equivalence (2.17) in the direction ($\Leftarrow$) follows from Corollary 1.5 and Lemma 2.5.

Now we prove equivalence (2.17) in the direction ($\Rightarrow$). Assume that $A_n > \alpha$ or $B_n > \beta$. Setting

$$\mathbb{D} \ni z \mapsto f_k(z) := \frac{(1-z)^{a_k}}{(1+z)^{b_k}}$$

for $k \in \mathbb{N}_n$ and

$$\mathbb{D} \ni z \mapsto f(z) := \prod_{k=1}^{n} f_k(z) \,,$$

we get $f_k \in K(a_k, b_k)$ for $k \in \mathbb{N}_n$, $f \in K_{a,b}$ and

$$f(z) = \frac{(1-z)^{A_n}}{(1+z)^{B_n}} \,.$$

Using (1.3) with r, θ_1 and θ_2 replaced by $\mathbb{N} \ni n \mapsto r(n) := 1 - 1/(2n^2)$, $\mathbb{N} \ni n \mapsto \theta_1(n) := 1/n$ and $\mathbb{N} \ni n \mapsto \theta_2(n) := 2\pi - 1/n$, respectively, we deduce that if $\alpha < A_n$, then for any $\beta \geq 0$, $f \notin K(\alpha, \beta)$. Using (1.3) with r, θ_1 and θ_2 replaced by $\mathbb{N} \ni n \mapsto r(n) := 1 - 1/(2n^2)$, $\mathbb{N} \ni n \mapsto \theta_1(n) := -\pi + 1/n$ and $\mathbb{N} \ni n \mapsto \theta_2(n) := \pi - 1/n$, respectively, we deduce that if $\beta < B_n$, then for any $\alpha \geq 0$, $f \notin K(\alpha, \beta)$. Therefore $f \notin K(\alpha, \beta)$. □

2.2. Connections with classes $\mathcal{S}$ and $\mathcal{C}$. In reference to Definition 2.1, we introduce the following subclasses of $\mathcal{H}$.

Definition 2.8. For $n \in \mathbb{N}$ and $a, b : \mathbb{N}_n \to \mathbb{R}_0^+$ we define

$$C_{a,b} = C_{(a_1,\ldots,a_n),(b_1,\ldots b_n)} := \{f \in \mathcal{H} : f' \in K_{a,b}\} \,. \tag{2.18}$$

and

$$\mathcal{D}_{\mathcal{C}} := \bigcup_{n\in\mathbb{N}} \bigcup_{a,b:\mathbb{N}_n \to \mathbb{R}_0^+} C_{a,b} \,.$$

Similarly to (2.2) and (2.3), we consider two inner operations in $\mathcal{D}_{\mathcal{C}}$ for all $m, n \in \mathbb{N}$, $a, b : \mathbb{N}_m \to \mathbb{R}_0^+$, $c, d : \mathbb{N}_n \to \mathbb{R}_0^+$ and $t \in \mathbb{R}$, defined as follows:

$$C_{a,b} \oplus C_{c,d} := \{h \in \mathcal{H} : h' \in K_{a,b} \oplus K_{c,d}\} \tag{2.19}$$

and

$$t \odot C_{a,b} := \{h \in \mathcal{H} : h' \in t \odot K_{a,b}\} \,. \tag{2.20}$$

We will show several auxiliary results, helpful for studying univalence of integral operators in the next section. To this aim we need the following result from [17].

Thorem B (Royster, 1965). *For any $w \in \mathbb{C}\setminus\{0\}$ the function $\mathbb{D} \ni z \mapsto f(z) := (1-z)^w$ is univalent if and only if $|w+1| \le 2$ or $|w-1| \le 2$.*

Using Royster's Theorem, Lemma 2.5 and Lemma 2.6, we can prove the following theorem.

Theorem 2.9. *For $n \in \mathbb{N}$ and $a, b : \mathbb{N}_n \to \mathbb{R}_0^+$ the following implications hold:*

$$(A_n \le 1 \text{ and } B_n \le 3) \Rightarrow C_{a,b} \subset \mathcal{C}\,, \tag{2.21}$$

$$(A_n > 1 \text{ or } B_n > 3) \Rightarrow C_{a,b} \not\subset \mathcal{S}\,. \tag{2.22}$$

Proof. Fix $n \in \mathbb{N}$ and $a, b : \mathbb{N}_n \to \mathbb{R}_0^+$. By Lemma 2.5 we get $K_{a,b} \subset K(A_n, B_n)$. If $A_n \le 1$ and $B_n \le 3$, then $f' \in K(1,3)$. It means that f is a close-to-convex function and it ends the proof of implication (2.21). By Lemma 2.6 we get $K(A_n, 0) \subset K_{a,b}$ and $K(0, B_n) \subset K_{a,b}$. Fix $A_n > 1$. Then setting $\mathbb{D} \ni z \mapsto f'(z) := (1-z)^{A_n}$, we conclude by Lemma 1.2 and (1.10) that $f' \in K(A_n, 0)\setminus K(1,3)$. Since

$$f(z) = \int_0^z (1-u)^{A_n} \mathrm{d}u = -\frac{1}{1+A_n} \cdot (1-z)^{1+A_n} + \frac{1}{1+A_n}$$

and $1 + A_n > 2$, from Theorem B we see that the function f is not univalent. That is $C_{a,b} \not\subset \mathcal{S}$.

Fix $B_n > 3$. Then setting $\mathbb{D} \ni z \mapsto f'(z) := (1-z)^{-B_n}$, by Lemma 1.2 and (1.10) we conclude that $f' \in K(0, B_n)\setminus K(1,3)$. Since

$$f(z) = \int_0^z 1/(1-u)^{B_n} \mathrm{d}u = -\frac{1}{1-B_n} \cdot (1-z)^{1-B_n} + \frac{1}{1-B_n}$$

and $1 - B_n < -2$, from Theorem B we see that the function f is not univalent. That is $C_{a,b} \not\subset \mathcal{S}$, which ends the proof of implication (2.22). □

Using equivalences (1.14) and (1.15), we can rewrite Theorem 2.9 in an equivalent form.

Theorem 2.10. *For $n \in \mathbb{N}$ and $a, b : \mathbb{N}_n \to \mathbb{R}_0^+$ the following implications hold:*

$$K(A_n, B_n) \subset K(1,3) \Rightarrow C_{a,b} \subset \mathcal{C}\,, \tag{2.23}$$

$$K(A_n, B_n) \not\subset K(1,3) \Rightarrow C_{a,b} \not\subset \mathcal{S}\,. \tag{2.24}$$

Remark 2.11. Let us notice that Theorems 2.9 and 2.10 imply the following conditions:

- for $n \in \mathbb{N}$ and $a, b : \mathbb{N}_n \to \mathbb{R}_0^+$,

$$C_{a,b} \subset \mathcal{S} \iff (A_n \le 1 \text{ and } B_n \le 3)\,,$$

$$C_{a,b} \subset \mathcal{S} \iff K(A_n, B_n) \subset K(1,3)\,,$$

- for $a, b : \mathbb{N}_1 \to \mathbb{R}_0^+$,

$$\begin{aligned}
&(a_1 \leq 1 \text{ and } b_1 \leq 3) \Rightarrow C_{a,b} \subset \mathcal{C}\,,\\
&(a_1 > 1 \text{ or } b_1 > 3) \Rightarrow C_{a,b} \not\subset \mathcal{S}\,,\\
&K_{a,b} = K(a_1, b_1) \subset K(1,3) \Rightarrow C_{a,b} \subset \mathcal{C}\,,\\
&K_{a,b} = K(a_1, b_1) \not\subset K(1,3) \Rightarrow C_{a,b} \not\subset \mathcal{S}\,,\\
&C_{a,b} \subset \mathcal{S} \iff (a_1 \leq 1 \text{ and } b_1 \leq 3)\,,\\
&C_{a,b} \subset \mathcal{S} \iff K_{a,b} = K(a_1, b_1) \subset K(1,3)\,.
\end{aligned}$$

The first two equivalences are necessary and sufficient conditions for univalence of all functions from classes $C_{a,b}$ which is the result directly related to classical Kaplan classes. Moreover, we can generalize these conditions to:

$$\begin{aligned}
&(a_1 \leq \alpha_0 \text{ and } b_1 \leq \beta_0) \Rightarrow K_{a,b} \subset K(\alpha_0, \beta_0)\,,\\
&(a_1 > \alpha_0 \text{ or } b_1 > \beta_0) \Rightarrow K_{a,b} \not\subset K(\alpha_0, \beta_0)
\end{aligned}$$

for arbitrarily fixed $\alpha_0, \beta_0 \geq 0$. Implications obtained in this way can be used to study some geometric properties described by parameters α_0 and β_0. For example by setting $\alpha_0 := 0$ and $\beta_0 := 2$, we get the necessary and sufficient conditions for convexity of functions from classes $C_{a,b}$.

3. The univalence of integral operators. In this section we apply the generalized Kaplan classes to study several problems in theory of univalence of integral operators. First we quote the definition of integral operator.

Definition 3.1. Let $n \in \mathbb{N}$ and $t : \mathbb{N}_n \to \mathbb{R}$. We consider an integral operator, which assigns to any $f_k \in \mathcal{H}_d$ for $k \in \mathbb{N}_n$ the following function

$$\mathbb{D} \ni z \mapsto h(z; (f_1, \ldots, f_n); (t_1, \ldots, t_n)) := \int_0^z \prod_{k=1}^n f_k^{t_k}(u)\mathrm{d}u \tag{3.1}$$

for $z \in \mathbb{D}$ and $F_n := (f_1, \ldots, f_n)$.

Remark 3.2. Let us notice that for $n \in \mathbb{N}$, $a, b : \mathbb{N}_n \to \mathbb{R}_0^+$, $t : \mathbb{N}_n \to \mathbb{R}$ and $f_k \in K(a_k, b_k)$ for $k \in \mathbb{N}_n$, study of univalence of the function (3.1) can be reduced to studying univalence of functions from classes $C_{a,b}$, i.e.

$$\begin{aligned}
&\Big\{h(\cdot; (f_1, \ldots, f_n); t) : \underset{k\in\mathbb{N}_n}{\forall} f_k \in K(a_k, b_k)\Big\}\\
&\qquad\qquad = (t_1 \odot C_{(a_1),(b_1)}) \oplus \ldots \oplus (t_n \odot C_{(a_n),(b_n)})
\end{aligned}$$

for $t : \mathbb{N}_n \mapsto \mathbb{R}$ and $a, b : \mathbb{N}_n \mapsto \mathbb{R}_0^+$.

To this aim we can appeal to Theorem 2.9 and 2.10. In particular we can simplify proofs or generalize results obtain in [10], [15], [16] and [17].

In this section we will use the following operators.

- If $n = 1$ and $f_1 = g/\,\mathrm{Id}$, where $g \in \mathcal{H}$, then

$$\mathbb{D} \ni z \mapsto h\left(z; \left(\frac{g}{\mathrm{Id}}\right); (t_2)\right) = \int_0^z \left(\frac{g(u)}{u}\right)^{t_2} \mathrm{d}u \tag{3.2}$$

is an integral operator of the first type (see [7, 9]).

- If $n = 1$ and $f_1 = f'$, where $f \in \mathcal{H}$, then

$$\mathbb{D} \ni z \mapsto h(z; (f'); (t_1)) = \int_0^z (f'(u))^{t_1} \mathrm{d}u \tag{3.3}$$

is an integral operator of the second type (see [7, 15]).

- If $n = 2$, $f_1 = f'$ and $f_2 = g/\,\mathrm{Id}$, where $f, g \in \mathcal{H}$, then

$$\mathbb{D} \ni z \mapsto h\left(z; \left(f', \frac{g}{\mathrm{Id}}\right); (t_1, t_2)\right) = \int_0^z (f'(u))^{t_1} \left(\frac{g(u)}{u}\right)^{t_2} \mathrm{d}u \tag{3.4}$$

is an integral operator studied in many papers (see [6]).

- If $n = 2$, $f_1 = f'$ and $f_2 = g'$, where $f, g \in \mathcal{H}$, then

$$\mathbb{D} \ni z \mapsto h\left(z; \left(f', g'\right); (t_1, t_2)\right) = \int_0^z (f'(u))^{t_1} \left(g'(u)\right)^{t_2} \mathrm{d}u \tag{3.5}$$

is an integral operator studied in many papers (see [14]).

- If $n = 2$, $f_1 = f/\,\mathrm{Id}$ and $f_2 = g/\,\mathrm{Id}$, where $f, g \in \mathcal{H}$, then

$$\mathbb{D} \ni z \mapsto h\left(z; \left(\frac{f}{\mathrm{Id}}, \frac{g}{\mathrm{Id}}\right); (t_1, t_2)\right) = \int_0^z \left(\frac{f(u)}{u}\right)^{t_1} \left(\frac{g(u)}{u}\right)^{t_2} \mathrm{d}u \tag{3.6}$$

is an integral operator studied in many papers (see [7]).

- If $n = 3$, $f_1 = f_1'$, $f_2 = f_2/\,\mathrm{Id}$ and $f_3 = f_3'$, where $f_1, f_2, f_3 \in \mathcal{H}$, then for every $z \in \mathbb{D}$

$$h\left(z; \left(f_1', \frac{f_2}{\mathrm{Id}}, f_3'\right); (t_1, t_2, t_3)\right) = \int_0^z (f_1'(u))^{t_1} \left(\frac{f_2(u)}{u}\right)^{t_2} (f_3'(u))^{t_3} \mathrm{d}u \tag{3.7}$$

is an integral operator introduced by us. If $t_1 = 0$ or $t_2 = 0$ or $t_3 = 0$, then (3.7) reduces to one of (3.2)–(3.5), classically studied. We study the operator given by (3.7) for functions $f_1 \in \mathcal{K}$, $f_2 \in \mathcal{S}^*$ and $f_3 \in \mathcal{C}$ obtaining the necessary and sufficient conditions for its univalence.

3.1. Generalization of univalence problem. For all $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$, $\alpha_1 + \beta_1 > 0$, $\alpha_2 + \beta_2 > 0$ and $k, n \in \{-1, 1\}$ we define

$$\frac{1}{0} := \lim_{\alpha \to 0^+} \frac{1}{\alpha} = +\infty\,,$$

$$d_1 := \begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix}, \qquad d_2 := \begin{vmatrix} \alpha_1 & \beta_2 \\ \beta_1 & \alpha_2 \end{vmatrix},$$

$$d_+ := n + k\,, \qquad d_- := n - k\,,$$

$$A_{k,n} := \left(\frac{-2\alpha_2(2+n) + 2\beta_2(2-n)}{(n+k)d_1 + (n-k)d_2}, \frac{-2\beta_1(2-k) + 2\alpha_1(2+k)}{(n+k)d_1 + (n-k)d_2} \right),$$

$$B_{k,n} := \left(\frac{d_+}{2} \min\left\{ \frac{4 - d_+}{2\alpha_1}, \frac{4 + d_+}{2\beta_1} \right\}, \frac{d_-}{2} \min\left\{ \frac{4 - d_-}{2\alpha_2}, \frac{4 + d_-}{2\beta_2} \right\} \right)$$

and we denote

$$A := \left\{ A_{k,n} : k, n \in \{-1, 1\} \text{ and } \frac{\alpha_1(1+2k) + \beta_1(1-2k)}{\alpha_2(1+2n) + \beta_2(1-2n)} < 0 \right\},$$

$$B := \{B_{k,n} : k, n \in \{-1, 1\}\}\,.$$

Lemma 3.3. *For all $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$, $\alpha_1 + \beta_1 > 0$, $\alpha_2 + \beta_2 > 0$ and $k, n \in \{-1, 1\}$ the condition $\overline{\overline{A}} \in \{0, 1, 2\}$ holds, where $\overline{\overline{A}}$ denotes the power of the set A.*

Proof. From the definition of A we see that $\overline{\overline{A}} \leq 4$. Assume that $\overline{\overline{A}} = 4$. Then the following system of inequalities holds:

$$\begin{cases} (3\alpha_1 - \beta_1)(3\alpha_2 - \beta_2) < 0\,, \\ (3\alpha_1 - \beta_1)(3\beta_2 - \alpha_2) < 0\,, \\ (3\beta_1 - \alpha_1)(3\alpha_2 - \beta_2) < 0\,, \\ (3\beta_1 - \alpha_1)(3\beta_2 - \alpha_2) < 0\,. \end{cases} \tag{3.8}$$

Adding the above inequalities by sides we get

$$4(\alpha_1 + \beta_1)(\alpha_2 + \beta_2) < 0\,,$$

which leads to the contradiction. Hence $\overline{\overline{A}} \leq 3$.

Now assume that $\overline{\overline{A}} = 3$. Then exactly three of the inequalities from (3.8) hold. Without loosing generality, we can assume that the first three inequalities from (3.8) hold. Multiplying them by sides we get

$$(3\alpha_1 - \beta_1)^2(3\alpha_2 - \beta_2)^2(3\beta_2 - \alpha_2)(3\beta_1 - \alpha_1) < 0\,,$$

from which we obtain $(3\beta_2 - \alpha_2)(3\beta_1 - \alpha_1) < 0$, which leads to the contradiction. Hence $\overline{\overline{A}} \leq 2$.

To prove that $\overline{\overline{A}}$ can be equal to 2, it is enough to set $\alpha_1 = \alpha_2 = 0$ and take arbitrary $\beta_1, \beta_2 > 0$. To prove that $\overline{\overline{A}}$ can be equal to 1, it is enough to set $\alpha_1 = 1$, $\alpha_2 = 5$, $\beta_1 = 3$ and $\beta_2 = 1$. To prove that $\overline{\overline{A}}$ can be equal to 0, it is enough to set $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$. □

Define the class $C(\alpha, \beta) := \{f \in \mathcal{H} : f' \in K(\alpha, \beta)\}$ for $\alpha, \beta \geq 0$. Now for the polygon given by

$$P := \operatorname{conv}(A \cup B)$$

we have the following theorem.

Theorem 3.4. *If $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$, $\alpha_1+\beta_1 > 0$, $\alpha_2+\beta_2 > 0$, $f \in K(\alpha_1, \beta_1)$, $g \in K(\alpha_2, \beta_2)$, $(t_1, t_2) \in P$ and $h(\cdot;(f,g);(t_1,t_2))$ is given by (3.1), then the function $h(\cdot;(f,g);(t_1,t_2)) \in \mathcal{C}$. Moreover, for each pair $(t_1, t_2) \notin P$ exist functions $f \in K(\alpha_1, \beta_1)$ and $g \in K(\alpha_2, \beta_2)$, such that the operator $h(\cdot;(f,g);(t_1,t_2)) \notin \mathcal{S}$.*

Proof. Assume that $t_1, t_2 \in \mathbb{R}$. Then

$$\begin{aligned}
&\{h(\cdot;(f,g);(t_1,t_2)) : f \in K(\alpha_1,\beta_1), g \in K(\alpha_2,\beta_2)\} \\
&\quad = t_1 \odot C(\alpha_1,\beta_1) \oplus t_2 \odot C(\alpha_2,\beta_2) \\
&\quad = C_{\left(\frac{\alpha_1+\beta_1}{2}|t_1| + \frac{\alpha_1-\beta_1}{2}t_1\right), \left(\frac{\alpha_1+\beta_1}{2}|t_1| - \frac{\alpha_1-\beta_1}{2}t_1\right)}.
\end{aligned}$$

By Theorem 2.9 we get the following system of inequalities

$$\begin{cases} \dfrac{\alpha_1+\beta_1}{2}|t_1| + \dfrac{\alpha_1-\beta_1}{2}t_1 + \dfrac{\alpha_2+\beta_2}{2}|t_2| + \dfrac{\alpha_2-\beta_2}{2}t_2 \leq 1\,, \\ \dfrac{\alpha_1+\beta_1}{2}|t_1| - \dfrac{\alpha_1-\beta_1}{2}t_1 + \dfrac{\alpha_2+\beta_2}{2}|t_2| - \dfrac{\alpha_2-\beta_2}{2}t_2 \leq 3\,. \end{cases} \tag{3.9}$$

Since (3.9) contains absolute values of t_1 and t_2, we should consider four cases. Now we assume that $t_1, t_2 \geq 0$. Then (3.9) takes the following form

$$\begin{cases} \alpha_1 t_1 + \alpha_2 t_2 \leq 1\,, \\ \beta_1 t_1 + \beta_2 t_2 \leq 3\,. \end{cases} \tag{3.10}$$

For $t_1 = 0$ we get

$$t_2 \leq \min\left\{\frac{1}{\alpha_2}, \frac{3}{\beta_2}\right\}$$

and for $t_2 = 0$ we get

$$t_1 \leq \min\left\{\frac{1}{\alpha_1}, \frac{3}{\beta_1}\right\}\ .$$

Hence

$$\left(0, \min\left\{\frac{1}{\alpha_2}, \frac{3}{\beta_2}\right\}\right) = B_{-1,1}$$

and

$$\left(\min\left\{\frac{1}{\alpha_1}, \frac{3}{\beta_1}\right\}, 0\right) = B_{1,1}$$

are the extremal points reached on axes by (3.10) for $t_1, t_2 \geq 0$. The system of equations

$$\left\{ \begin{array}{l} \alpha_1 t_1 + \alpha_2 t_2 = 1\,, \\ \beta_1 t_1 + \beta_2 t_2 = 3 \end{array} \right. \tag{3.11}$$

has a single solution for $t_1, t_2 > 0$ if and only if $(3\alpha_1 - \beta_1)(3\alpha_2 - \beta_2) < 0$. This solution has the form

$$(t_1, t_2) = \left(\frac{\beta_2 - 3\alpha_2}{\alpha_1\beta_2 - \alpha_2\beta_1}, \frac{3\alpha_1 - \beta_1}{\alpha_1\beta_2 - \alpha_2\beta_1} \right) = A_{1,1}\,, \tag{3.12}$$

which ends the proof of the case $t_1, t_2 \geq 0$. Three remaining cases can be proved analogously and we get the following points

$$\begin{array}{llll} B_{-1,1}, & B_{-1,-1}, & A_{-1,1}, & \text{for} \quad t_1 \leq 0 \text{ and } t_2 \geq 0\,, \\ B_{-1,-1}, & B_{1,-1}, & A_{-1,-1}, & \text{for} \quad t_1 \leq 0 \text{ and } t_2 \leq 0\,, \\ B_{1,-1}, & B_{1,1}, & A_{1,-1}, & \text{for} \quad t_1 \geq 0 \text{ and } t_2 \leq 0\,. \end{array}$$

□

Remark 3.5. Let us notice that the polygon P can be a convex hull of four, five or six points, because $\overline{\overline{A}} \leq 2$ and $\overline{\overline{B}} = 4$.

Remark 3.6. Let us notice that if $\alpha_1 = \beta_1 = 0$ or $\alpha_2 = \beta_2 = 0$, then $f \in K(0,0)$ or $g \in K(0,0)$. In this case, from the implication (1.8), we see that the operator $h(\cdot;(f,g);(t_1,t_2))$ given by (3.1) is reduced to $h(\cdot;(g);(t_2))$ or $h(\cdot;(f);(t_1))$, respectively.

Theorem 3.4 allows us to achieve the necessary and sufficient condition for univalence of an operator h for any functions from Kaplan classes. Now we show the example of use of this condition.

Example 3.7. Consider the function $h(\cdot;(f,g);(t_1,t_2))$ given by (3.4) for $f \in K(1/4, 1/2)$ and $g \in K(1, 1/5)$. From Theorem 3.4 we get

$$A \cup B = \left\{ \Big(4, 0\Big), \Big(0, 1\Big), \Big(-2, 0\Big), \Big(0, -3\Big), \left(-\frac{8}{9}, -\frac{25}{9}\right), \left(\frac{8}{3}, -\frac{5}{3}\right) \right\}.$$

Let us notice that this example is important, because we cannot use the Sheil-Small factorization theorem when $(\alpha_1 - \beta_1)(\alpha_2 - \beta_2) < 0$ as it is in this case.

3.2. Simplified proofs. Methods used in Theorem 3.4 can also be applied in proofs of many theorems with integral operators given by (3.2)–(3.6) for classes $\mathcal{K}$, $\mathcal{S}^*$ and $\mathcal{C}$. This allows us to simplify proofs. As an example we give two alternative proofs of the following theorem from [10].

Theorem 3.8 (Kim and Merkes, 1974). *Assume that*

$$P_1 = \operatorname{conv}\left(\left\{\left(-\frac{1}{2},\frac{3}{2}\right),\left(0,\frac{3}{2}\right),\left(\frac{3}{2},0\right),\left(\frac{3}{2},-\frac{1}{2}\right),\left(0,-\frac{1}{2}\right),\left(-\frac{1}{2},0\right)\right\}\right),$$

$f\in\mathcal{K}$ and $g\in\mathcal{S}^*$. Then the operator $h(\cdot;(f',g/\mathrm{Id});(t_1,t_2))$ given by (3.4) *is in $\mathcal{C}$ for each pair $(t_1,t_2)\in P_1$. Moreover, for each pair $(t_1,t_2)\notin P_1$ there exist functions $f\in\mathcal{K}$, and $g\in\mathcal{S}^*$ such that operator $h(\cdot;(f',g/\mathrm{Id});(t_1,t_2))$ is not univalent in $\mathbb{D}$.*

Proof. Method I. Assume that $t_1,t_2\in\mathbb{R}$. Then

$$\left\{h\left(\cdot;\left(f',\frac{g}{\mathrm{Id}}\right);(t_1,t_2)\right) : f\in\mathcal{K}, g\in\mathcal{S}^*\right\} = t_1\odot C(0,2)\oplus t_2\odot C(0,2)$$
$$= C_{\left(\frac{|t_1|+t_1}{2}\cdot 0+\frac{|t_1|-t_1}{2}\cdot 2+\frac{|t_2|+t_2}{2}\cdot 0+\frac{|t_2|-t_2}{2}\cdot 2\right),\left(\frac{|t_1|+t_1}{2}\cdot 2+\frac{|t_1|-t_1}{2}\cdot 0+\frac{|t_2|+t_2}{2}\cdot 2+\frac{|t_2|-t_2}{2}\cdot 0\right)}$$
$$= C_{(|t_1|-t_1+|t_2|-t_2),(|t_1|+t_1+|t_2|+t_2)}\,.$$

From Theorem 2.9 we get the following system of inequalities

$$\begin{cases} |t_1|-t_1+|t_2|-t_2\le 1\,,\\ |t_1|+t_1+|t_2|+t_2\le 3\,,\end{cases}$$

from which $(t_1,t_2)\in P_1$.

Method II. We know that $f', g/\mathrm{Id}\in K(0,2)$. Therefore for $k,n\in\{-1,1\}$ we get

$$A_{k,n} = \left(\frac{2-n}{k-n},\frac{k-2}{k-n}\right)$$

and

$$B_{k,n} = \left(\frac{(n+k)(4+n+k)}{8},\frac{(n-k)(4+n-k)}{8}\right),$$

from which we obtain

$$A = \left\{\left(\frac{3}{2},-\frac{1}{2}\right),\left(-\frac{1}{2},\frac{3}{2}\right)\right\}$$

and

$$B = \left\{\left(\frac{3}{2},0\right),\left(-\frac{1}{2},0\right),\left(0,-\frac{1}{2}\right),\left(0,\frac{3}{2}\right)\right\}.$$

Setting $P_1 := \operatorname{conv}(A\cup B)$, from Theorem 3.4 we get

$$h\left(\cdot;f',\left(\frac{g}{\mathrm{Id}}\right);(t_1,t_2)\right)\in\mathcal{C}$$

for all $f\in\mathcal{K}$, $g\in\mathcal{S}^*$ and $(t_1,t_2)\in P_1$. Moreover, there exist functions $f\in\mathcal{K}$ and $g\in\mathcal{S}^*$ such that

$$h\left(\cdot;f',\left(\frac{g}{\mathrm{Id}}\right);(t_1,t_2)\right)\notin\mathcal{S}$$

for $(t_1,t_2)\notin P_1$. □

Results from [4], [5], [7], [10] and [15] are presented here as simple corollaries of Theorem 3.4.

Corollary 3.9. *If $f \in \mathcal{K}$, $t_1 \in [-1/2;3/2]$ and $h(\cdot;(f');(t_1))$ is given by* (3.2)*, then $h(\cdot;(f');(t_1)) \in \mathcal{C}$. Moreover, if $t_1 \notin [-1/2;3/2]$, then there exists $f \in \mathcal{K}$ such that $h(\cdot;(f');(t_1))$ given by* (3.2) *is not univalent.*

Corollary 3.10. *If $f \in \mathcal{C}$, $t_1 \in [-1/3;1]$ and $h(\cdot;(f');(t_1))$ is given by* (3.2)*, then $h(\cdot;(f');(t_1)) \in \mathcal{C}$. Moreover, if $t_1 \notin [-1/3;1]$, then there exists $f \in \mathcal{C}$ such that $h(\cdot;(f');(t_1))$ given by* (3.2) *is not univalent.*

Corollary 3.11. *If $g \in \mathcal{S}^*$, $t_2 \in [-1/2;3/2]$ and $h(\cdot;(g/\mathrm{Id});(t_2))$ is given by* (3.3)*, then $h(\cdot;(g/\mathrm{Id});(t_2)) \in \mathcal{C}$. Moreover, if $t_2 \notin [-1/2;3/2]$, then there exists $g \in \mathcal{S}^*$ such that $h(\cdot;(g/\mathrm{Id});(t_2))$ given by* (3.3) *is not univalent.*

Corollary 3.12. *Assume that*

$$P_2 = \mathrm{conv}\left(\left\{\left(-\frac{1}{3},\frac{4}{3}\right),\left(0,\frac{3}{2}\right),\left(1,0\right),\left(0,-\frac{1}{2}\right),\left(-\frac{1}{3},0\right)\right\}\right),$$

$f \in \mathcal{C}$ and $g \in \mathcal{S}^*$. Then the operator $h(\cdot;(f',g/\mathrm{Id});(t_1,t_2))$ given by (3.4) *is in $\mathcal{C}$ for each pair $(t_1,t_2) \in P_2$. Moreover, for each pair $(t_1,t_2) \notin P_2$ there exist functions $f \in \mathcal{C}$ and $g \in \mathcal{S}^*$ such that $h(\cdot;(f',g/\mathrm{Id});(t_1,t_2))$ given by* (3.4) *is not univalent.*

Corollary 3.13. *Assume that*

$$P_3 = \mathrm{conv}\left(\left\{\left(-\frac{1}{2},\frac{3}{2}\right),\left(0,\frac{3}{2}\right),\left(\frac{3}{2},0\right),\left(\frac{3}{2},-\frac{1}{2}\right),\left(0,-\frac{1}{2}\right),\left(-\frac{1}{2},0\right)\right\}\right)$$

and $f,g \in \mathcal{K}$. Then the operator $h(\cdot;(f',g');(t_1,t_2))$ given by (3.5) *is in $\mathcal{C}$ for each pair $(t_1,t_2) \in P_3$. Moreover, for each pair $(t_1,t_2) \notin P_3$ there exist functions $f \in \mathcal{K}$, and $g \in \mathcal{K}$ such that $h(\cdot;(f',g');(t_1,t_2))$ given by* (3.5) *is not univalent.*

Corollary 3.14. *Assume that*

$$P_4 = \mathrm{conv}\left(\left\{\left(-\frac{1}{3},\frac{4}{3}\right),\left(0,\frac{3}{2}\right),\left(1,0\right),\left(0,-\frac{1}{2}\right),\left(-\frac{1}{3},0\right)\right\}\right),$$

$f \in \mathcal{K}$ and $g \in \mathcal{C}$. Then the operator $h(\cdot;(f',g');(t_1,t_2))$ given by (3.5) *is in $\mathcal{C}$ for each pair $(t_1,t_2) \in P_4$. Moreover, for each pair $(t_1,t_2) \notin P_4$ there exist functions $f \in \mathcal{K}$, and $g \in \mathcal{C}$ such that $h(\cdot;(f',g');(t_1,t_2))$ given by* (3.5) *is not univalent.*

Corollary 3.15. *Assume that*

$$P_5 = \mathrm{conv}\left(\left\{\left(-\frac{1}{3},0\right),\left(0,1\right),\left(1,0\right),\left(0,-\frac{1}{3}\right)\right\}\right),$$

$f \in \mathcal{C}$ and $g \in \mathcal{C}$. Then the operator $h(\cdot;(f',g');(t_1,t_2))$ given by (3.5) *is in $\mathcal{C}$ for each pair $(t_1,t_2) \in P_5$. Moreover, for each pair $(t_1,t_2) \notin P_5$ there exist functions $f \in \mathcal{C}$, and $g \in \mathcal{C}$ such that $h(\cdot;(f',g');(t_1,t_2))$ given by* (3.5) *is not univalent.*

Corollary 3.16. *Assume that*

$$P_6 = \operatorname{conv}\left(\left\{\left(-\frac{1}{2},\frac{3}{2}\right),\left(0,\frac{3}{2}\right),\left(\frac{3}{2},0\right),\left(\frac{3}{2},-\frac{1}{2}\right),\left(0,-\frac{1}{2}\right),\left(-\frac{1}{2},0\right)\right\}\right)$$

and $f,g\in\mathcal{S}^*$*. Then the operator* $h(\cdot;(f/\mathrm{Id},g/\mathrm{Id});(t_1,t_2))$ *given by* (3.6) *is in* $\mathcal{C}$ *for each pair* $(t_1,t_2)\in P_6$*. Moreover, for each pair* $(t_1,t_2)\notin P_6$ *there exist functions* $f\in\mathcal{S}^*$*, and* $g\in\mathcal{S}^*$ *such that* $h(\cdot;(f/\mathrm{Id},g/\mathrm{Id});(t_1,t_2))$ *given by* (3.6) *is not univalent.*

Let us notice that $P_2 = P_4$ and $P_3 = P_6$. This fact follows also from Alexander's Theorem.

Causey and Reade considered the classes B_m for $m > 0$ partly referring to the Kaplan classes (see [4, p. 9]). However, the nature of these classes caused them to obtain very limited results. Firstly, they applied only functions f of specific classes under the integral

$$\int_0^z (f'(t))^\alpha \left(\frac{f(t)}{t}\right)^{1-\alpha} \mathrm{d}t$$

and secondly, they received only a sufficient condition of univalence (see [4]).

3.3. Extension of univalence problem. Methods described in this work allow us to expand the necessary and sufficient condition for univalence of the operator h to operators of greater dimensions. Now we present the following theorem for the operator given by (3.7).

Theorem 3.17. *Assume that*

$$\begin{aligned}P_7 = \operatorname{conv}\Bigg(\Bigg\{&\left(0,0,1\right),\left(0,-\frac{1}{2},0\right),\left(\frac{3}{2},-\frac{1}{2},0\right),\left(\frac{4}{3},0,-\frac{1}{3}\right),\left(\frac{3}{2},0,0\right),\\&\left(0,0,-\frac{1}{3}\right),\left(-\frac{1}{2},0,0\right),\left(-\frac{1}{2},\frac{3}{2},0\right),\left(0,\frac{3}{2},0\right),\left(0,\frac{4}{3},-\frac{1}{3}\right)\Bigg\}\Bigg),\end{aligned}$$

$f_1\in\mathcal{K}$*,* $f_2\in\mathcal{S}^*$ *and* $f_3\in\mathcal{C}$*. Then for each* $(t_1,t_2,t_3)\in P_7$ *the operator* $h(\cdot;(f_1',f_2/\mathrm{Id},f_3');(t_1,t_2,t_3))$ *given by* (3.7) *is in* $\mathcal{C}$*. Moreover, for each* $(t_1,t_2,t_3)\notin P_7$ *there exist functions* $f_1\in\mathcal{K}$*,* $f_2\in\mathcal{S}^*$ *and* $f_3\in\mathcal{C}$ *such that* $h(\cdot;(f_1',f_2/\mathrm{Id},f_3');(t_1,t_2,t_3))$ *given by* (3.7) *is not univalent.*

Proof. Assume that $t_1,t_2,t_3\in\mathbb{R}$. Then we have

$$\begin{aligned}&\left\{h\left(\cdot;\left(f_1',\frac{f_2}{\mathrm{Id}},f_3'\right);(t_1,t_2,t_3)\right) : f_1\in\mathcal{K}, f_2\in\mathcal{S}^*, f_3\in\mathcal{C}\right\}\\&\quad= t_1\odot C(0,2)\oplus t_2\odot C(0,2)\oplus t_3\odot C(1,3)\\&\quad= C_{(|t_1|-t_1+|t_2|-t_2+2|t_3|-t_3),(|t_1|+t_1+|t_2|+t_2+2|t_3|+t_3)}\,.\end{aligned}$$

By Theorem 2.9 we get the following system of inequalities

$$\begin{cases} |t_1| - t_1 + |t_2| - t_2 + 2|t_3| - t_3 \leq 1\,, \\ |t_1| + t_1 + |t_2| + t_2 + 2|t_3| + t_3 \leq 3\,, \end{cases}$$

from which $(t_1, t_2, t_3) \in P_7$. □

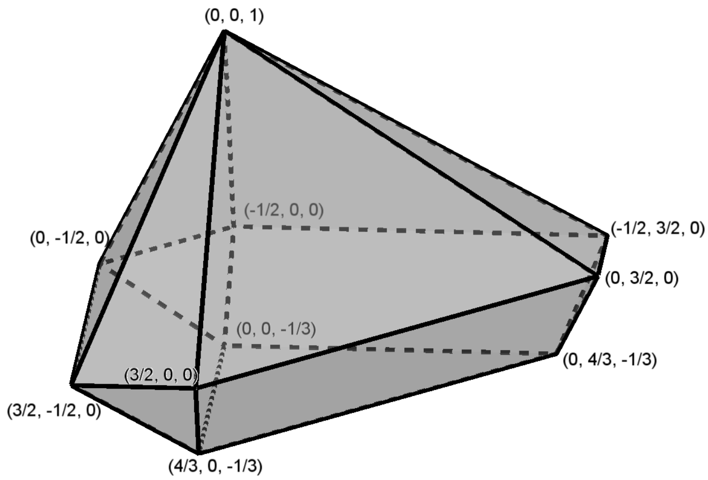

FIGURE 1. The set P_7.

Now we present a necessary and sufficient condition for univalence of any operator h given by (3.1).

Theorem 3.18. *Assume that $n \in \mathbb{N}$, $t : \mathbb{N}_n \to \mathbb{R}$, $f_k \in K(\alpha_k, \beta_k)$ for $\alpha_k, \beta_k \geq 0$ and $k \in \mathbb{N}_n$. Let $P_8 \subset \mathbb{R}^n$ such that $x = (x_1, x_2, \ldots, x_n) \in P_8$ if and only if*

$$\begin{cases} \frac{\alpha_1+\beta_1}{2}|x_1| + \frac{\alpha_1-\beta_1}{2}x_1 + \cdots + \frac{\alpha_n+\beta_n}{2}|x_n| + \frac{\alpha_n-\beta_n}{2}x_n \leq 1\,, \\ \frac{\alpha_1+\beta_1}{2}|x_1| - \frac{\alpha_1-\beta_1}{2}x_1 + \cdots + \frac{\alpha_n+\beta_n}{2}|x_n| - \frac{\alpha_n-\beta_n}{2}x_n \leq 3\,. \end{cases}$$

Then for each $t \in P_8$ the operator $h(\cdot; F_n; t)$ given by (3.1) is in $\mathcal{C}$. Moreover, for each $t \notin P_8$ there exist functions $f_k \in K(\alpha_k, \beta_k)$ for $k \in \mathbb{N}_n$ such that $h(\cdot; F_n; t)$ given by (3.1) is not univalent.

Proof. Assume that $n \in \mathbb{N}$, $t : \mathbb{N}_n \to \mathbb{R}$. Then analogously as in the previous case we have

$$\left\{ h\left(\cdot; (f_1, f_2 \ldots, f_n); (t_1, t_2, \ldots, t_n)\right) : \underset{k \in \mathbb{N}_n}{\forall} f_k \in K(\alpha_k, \beta_k) \right\}$$
$$= t_1 \odot C(\alpha_1, \beta_1) \oplus t_2 \odot C(\alpha_2, \beta_2) \oplus \cdots \oplus t_n \odot C(\alpha_n, \beta_n)\,.$$

By Theorem 2.9 we get the following system of inequalities

$$\begin{cases} \dfrac{\alpha_1+\beta_1}{2}|t_1| + \dfrac{\alpha_1-\beta_1}{2}t_1 + \cdots + \dfrac{\alpha_n+\beta_n}{2}|t_n| + \dfrac{\alpha_n-\beta_n}{2}t_n \le 1\,, \\ \dfrac{\alpha_1+\beta_1}{2}|t_1| - \dfrac{\alpha_1-\beta_1}{2}t_1 + \cdots + \dfrac{\alpha_n+\beta_n}{2}|t_n| - \dfrac{\alpha_n-\beta_n}{2}t_n \le 3\,, \end{cases}$$

from which $t \in P_8$. □

Conflict of Interest: The authors declare that they have no conflict of interest.

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