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On Dirichlet type spaces on the unit ball of $\mathbb{C}^n$

Dedicated to the memory of Professor Jan G. Krzyż

Abstract. In this paper we discuss characterizations of Dirichlet type spaces on the unit ball of $\mathbb{C}^n$ obtained by P. Hu and W. Zhang [2], and S. Li [4].

1. Introduction. Let $B = B_n = \{z \in \mathbb{C}^n : |z| < 1\}$ be the open unit ball in $\mathbb{C}^n$ $(n \geq 1)$ and $S = S_n = \{z \in \mathbb{C}^n : |z| = 1\}$ be its boundary. Let dv denote the normalized Lebesgue measure on B, i.e. $dv(B) = 1$ and $d\sigma$ denote the normalized rotation invariant measure on S, i.e. $d\sigma(S) = 1$. The class of holomorphic functions in B will be denoted by $H(B)$. Let f be in $H(B)$ with Taylor expansion $f(z) = \sum_{|\alpha|\geq 0} a_\alpha z^\alpha$. For $p \in \mathbb{R}$, f is said to be in the Dirichlet type space D_p provided that

$$\sum_{|\alpha|\geq 0} (n + |\alpha|)^p |a_\alpha|^2 \omega_\alpha < \infty,$$

where

$$\omega_\alpha = \int_S |\zeta^\alpha|^2 d\sigma(\zeta) = \frac{(n-1)!\alpha!}{(n+|\alpha|-1)!}$$

and $\alpha = (\alpha_1, \ldots, \alpha_n)$ is n-tuple index, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $\alpha! = \alpha_1! \cdot \ldots \cdot \alpha_n!$, $z^\alpha = z_1^{\alpha_1} \cdot \ldots \cdot z_n^{\alpha_n}$. In [2] Hu and Zhang gave a BMO-type characterization of Dirichlet type spaces. Using this characterization, S. Li [4] obtained

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a description of the spaces D_p which is cited below. For $f \in H(B)$, let

$$Q_f(z) = \sup \left\{ \frac{|\langle \nabla f(z), \overline{w} \rangle|}{(H_z(w,w))^{\frac{1}{2}}} : 0 \neq w \in \mathbb{C}^n \right\},$$

where $\nabla f(z) = \left(\frac{\partial f}{\partial z_1}(z), \ldots, \frac{\partial f}{\partial z_n}(z) \right)$ is a holomorphic gradient of f and

$$H_z(w,w) = \frac{n+1}{2} \frac{(1-|z|^2)|w|^2 + |\langle w, z \rangle|^2}{(1-|z|^2)^2}.$$

Theorem A ([4]). *Suppose $-1 < p < n$. Then $f \in D_p$ if and only if*

$$\int_B Q_f^2(z)(1-|z|^2)^{-p-1} dv(z) < \infty. \tag{1}$$

However, the function $f_1(z) = z_1 \in D_p$, $p \in \mathbb{R}$, shows that this theorem is not true in the case when $p \geq 2$. Indeed, since (see [7], p. 80)

$$Q_f(z) \approx \left[(1-|z|^2)(|\nabla f(z)|^2 - |Rf(z)|^2) \right]^{\frac{1}{2}}, \tag{2}$$

we get

$$\begin{aligned} \int_B Q_{f_1}^2(z)(1-|z|^2)^{-p-1} dv(z) &\approx \int_B (1-|z_1|^2)(1-|z|^2)^{-p} dv(z) \\ &> \int_B (1-|z|^2)^{-p+1} dv(z) \end{aligned}$$

and the last integral diverges for $p \geq 2$. Throughout this paper we use the symbol $\approx$ to mean comparable.

On the other hand, it follows from Lemma 3 in [3] that for $p < 0$, $f \in D_p$ if and only if condition (1) is satisfied.

We also note that Dirichlet type spaces in one-dimensional setting were discussed in [1] and [5]. Due to results obtained in these papers we know that in the case when $n = 1$; for all $p \leq 1$

$$\begin{aligned} f \in D_p \Leftrightarrow & \int_B \int_B \frac{|f(z) - f(w)|^2}{|1 - \langle w, z \rangle|^{n+2+\delta+\tau+p}} (1-|z|^2)^\delta (1-|w|^2)^\tau dv(z) dv(w) \\ & \approx \int_B |f'(z)|^2 (1-|z|^2)^{1-p} dv(z) < \infty, \end{aligned}$$

where $\delta, \tau > -1$ and $\min(\delta, \tau) + p > -1$.

The proof of Theorem A is based on a theorem in ([2], p. 454). In the proof of the latter the authors used the following.

Lemma B ([2], p. 455). *Let α and γ be n-tuple indices, and let j be a nonnegative integer. Then*

$$\sum_{|\gamma|=j} \frac{(\gamma+\alpha)!}{\gamma! \alpha!} = \frac{(|\gamma|+|\alpha|)!}{|\gamma|!|\alpha|!} = \frac{(|\alpha|+j)!}{|\alpha|! j!}. \tag{3}$$

Unfortunately this lemma is incorrect for $n > 1$ as the following example shows.

Let $\alpha = (1, 2)$. If $|\gamma| = 3$ then $\gamma \in \{(3, 0), (2, 1), (1, 2), (0, 3)\}$. In this case the left-hand side of equality (3) equals $35 \neq \frac{6!}{3!3!} = 20$.

In this paper we prove a correct version of the theorem given in [2]. It is the following.

Theorem 1. *Let* $f(z) = \sum_{|\alpha|\geq 0} a_\alpha z^\alpha \in H(B)$, $p < 1$, $\delta, \tau > -1$, $\min(\delta, \tau) + p > -1$, *then*

$$\begin{aligned} &\int_B \int_B \frac{|f(z) - f(w)|^2}{|1 - \langle w, z\rangle|^{n+2+\delta+\tau+p}} (1 - |z|^2)^\delta (1 - |w|^2)^\tau dv(z) dv(w) \\ &\qquad \approx \sum_{|\alpha|>0} (n + |\alpha|)^p |a_\alpha|^2 \omega_\alpha. \end{aligned} \tag{4}$$

In the proof of Theorem 1 we use the method introduced by R. Rochberg and Z. J. Wu in [5] and developed by J. H. Shi and P. Y. Hu in [3].

We have already mentioned that in the case when $n = 1$ and $p = 1$ Theorem 1 holds. Surprisingly, the situation is different for $n > 1$. More exactly, we show the following.

Theorem 2. *If* $n > 1$ *and* f *is a nonconstant holomorphic function in the unit ball* B *of* $\mathbb{C}^n$, *then*

$$I = \int_B \int_B \frac{|f(z) - f(w)|^2}{|1 - \langle w, z\rangle|^{n+3+\delta+\tau}} (1 - |z|^2)^\delta (1 - |w|^2)^\tau dv(z) dv(w) = +\infty$$

for all $\delta, \tau > -1$.

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2. The proofs of Theorems 1 and 2. We start with the following improved version of Lemma 2 in [2].

Lemma 1. *Let* α *and* γ *be* n*-tuple indices, and let* j *be a nonnegative integer. Then*

$$\sum_{|\gamma|=j} \frac{(\gamma + \alpha)!}{\gamma!\alpha!} = \frac{(|\gamma| + |\alpha| + n - 1)!}{|\gamma|!(|\alpha| + n - 1)!} = \frac{(|\alpha| + n + j - 1)!}{(|\alpha| + n - 1)!j!}. \tag{5}$$

Proof. We show first that for nonnegative integers x, y,

$$\sum_{k=0}^{j} \binom{x + k}{k} \binom{y + j - k}{j - k} = \binom{x + y + j + 1}{j}. \tag{6}$$

We will use mathematical induction for j. It is clear that for $j = 1$ the formula is valid. Now suppose that the formula holds for an arbitrarily chosen j. Multiplying both sides of equality (6) by $j!$, we get

$$\prod_{s=1}^{j}(y+s)+\sum_{k=1}^{j-1}\binom{j}{k}\prod_{s=1}^{k}(x+s)\prod_{r=1}^{j-k}(y+r)+\prod_{s=1}^{j}(x+s)$$
$$=\prod_{s=1}^{j}(x+y+1+s).$$

Next using the equality $\binom{n+1}{k}=\binom{n}{k}+\binom{n}{k-1}$, we obtain

$$(j+1)!\cdot\sum_{k=0}^{j+1}\binom{x+k}{k}\binom{y+j+1-k}{j+1-k}$$
$$=\prod_{s=1}^{j+1}(y+s)+\sum_{k=1}^{j}\binom{j}{k}\prod_{s=1}^{k}(x+s)\prod_{r=1}^{j+1-k}(y+r)$$
$$+\sum_{k=1}^{j}\binom{j}{k-1}\prod_{s=1}^{k}(x+s)\prod_{r=1}^{j+1-k}(y+r)+\prod_{s=1}^{j+1}(x+s)$$
$$=(y+1)\left[\prod_{s=1}^{j}(y+1+s)+\sum_{k=1}^{j-1}\binom{j}{k}\prod_{s=1}^{k}(x+s)\prod_{r=1}^{j-k}(y+1+r)+\prod_{s=1}^{j}(x+s)\right]$$
$$+(x+1)\left[\prod_{s=1}^{j}(y+s)+\sum_{k=1}^{j-1}\binom{j}{k}\prod_{s=1}^{k}(x+1+s)\prod_{r=1}^{j-k}(y+r)+\prod_{s=1}^{j}(x+1+s)\right]$$
$$=\prod_{s=1}^{j+1}(x+y+1+s)=(j+1)!\cdot\binom{x+y+j+2}{j+1}.$$

Now we will prove equality (5) by induction for n. For $n=1$ formula is valid for every nonnegative integer j. Assume that the equality is valid for an arbitrarily chosen n. Let $\gamma=(\gamma_1,\gamma')$ and $\alpha=(\alpha_1,\alpha')$ be the $(n+1)$-tuple indices where γ_1,α_1 are nonnegative integers. Then by (6),

$$\sum_{|\gamma|=j}\frac{(\gamma+\alpha)!}{\gamma!\alpha!}=\sum_{\gamma_1=1}^{j}\frac{(\gamma_1+\alpha_1)!}{\gamma_1!\alpha_1!}\cdot\sum_{|\gamma'|=j-\gamma_1}\frac{(\gamma'+\alpha')!}{\gamma'!\alpha'!}$$
$$=\sum_{\gamma_1=1}^{j}\frac{(\gamma_1+\alpha_1)!}{\gamma_1!\alpha_1!}\cdot\frac{(|\gamma'|+|\alpha'|+n-1)!}{|\gamma'|!(|\alpha'|+n-1)!}$$
$$=\sum_{\gamma_1=1}^{j}\binom{\alpha_1+\gamma_1}{\gamma_1}\binom{|\alpha'|+n-1+j-\gamma_1}{j-\gamma_1}=\binom{|\alpha|+n+j}{j}.$$

This completes the proof of formula (5). □

We will also need the following lemmas for proving Theorem 1.

Lemma C ([5]). *For $x, y > 0$, the Gamma and Beta functions are defined as*

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \qquad B(x,y) = \int_0^1 r^{x-1}(1-r)^{y-1}dr.$$

For fixed x and y, we have for any natural numbers j and k

$$\frac{\Gamma(j+x)}{\Gamma(j+y)} \approx (j+1)^{x-y}, \qquad B(k,y) \approx k^{-y}; \tag{7}$$

$$B(j+1+x,y) - B(j+k+1+x,y) \approx (j+1)^{-y} - (j+k+1)^{-y}. \tag{8}$$

Here "$\approx$" is independent of j and k.

Lemma D ([5]). *For $n > 1$, $p < 1$ and $p + \delta > -1$, we have*

$$\int_0^\infty t^{p+\delta}(t+1)^{\frac{p-n}{2}-\delta-1}\left[(t+1)^{\frac{n-p}{2}} - t^{\frac{n-p}{2}}\right]dt < \infty. \tag{9}$$

Proof of Theorem 1. It follows from ([5], p. 109) that without loss of generality we can assume that $\delta = \tau$. Put $2\beta = n + p + 2\delta + 2$ and

$$I = \int_B\int_B \frac{|f(z)-f(w)|^2}{|1-\langle w,z\rangle|^{2\beta}}(1-|z|^2)^\delta(1-|w|^2)^\tau dv(z)dv(w).$$

We start with the following formula given in ([2], p. 459)

$$\begin{aligned} I \approx \sum_{k=1}^\infty \sum_{|\alpha|=k} |a_\alpha|^2\omega_\alpha \sum_{j=0}^\infty & \frac{\Gamma(k+n)}{\Gamma(n)\Gamma(j+n+\delta+1)} \\ \times \Bigg(& \frac{\Gamma^2(j+\beta)}{\Gamma(j+k+n+\delta+1)} \sum_{|\gamma|=j} \frac{(\alpha+\gamma)!}{\gamma!\alpha!} \\ & - 2\frac{\Gamma(j+\beta)\Gamma(j-k+\beta)}{\Gamma(j+n+\delta+1)} \sum_{|\gamma|=j} \frac{\chi(\gamma-\alpha)\gamma!}{(\gamma-\alpha)!\alpha!} \\ & + \frac{\Gamma^2(j-k+\beta)}{\Gamma(j-k+n+\delta+1)} \sum_{|\gamma|=j} \frac{\chi(\gamma-\alpha)\gamma!}{(\gamma-\alpha)!\alpha!}\Bigg). \end{aligned}$$

Using (5), we obtain

$$
\begin{aligned}
I &\approx \sum_{k=1}^{\infty}\sum_{|\alpha|=k}|a_\alpha|^2\omega_\alpha\sum_{j=0}^{\infty}\frac{\Gamma(k+n)}{\Gamma(n)\Gamma(j+n+\delta+1)}\\
&\quad\times\Bigg(\frac{\Gamma^2(j+\beta)}{\Gamma(j+k+n+\delta+1)}\frac{\Gamma(j+k+n)}{\Gamma(k+n)\Gamma(j+1)}\\
&\qquad-2\frac{\chi(j-k)\Gamma(j+\beta)\Gamma(j-k+\beta)\Gamma(j+n)}{\Gamma(j+n+\delta+1)\Gamma(k+n)\Gamma(j-k+1)}\\
&\qquad+\frac{\chi(j-k)\Gamma^2(j-k+\beta)\Gamma(j+n)}{\Gamma(j-k+n+\delta+1)\Gamma(k+n)\Gamma(j-k+1)}\Bigg)\\
&\approx \sum_{k=1}^{\infty}\sum_{|\alpha|=k}|a_\alpha|^2\omega_\alpha\sum_{j=0}^{\infty}B(j+1,n+\delta)\\
&\quad\times\Bigg(B(j+k+1,n+\delta)\frac{\Gamma^2(j+\beta)\Gamma(j+k+n)}{\Gamma(j+k+1)\Gamma^2(j+1)}\\
&\qquad-2\chi(j-k)B(j+1,n+\delta)\frac{\Gamma(j+\beta)\Gamma(j-k+\beta)\Gamma(j+n)}{\Gamma(j-k+1)\Gamma^2(j+1)}\\
&\qquad+\chi(j-k)B(j-k+1,n+\delta)\frac{\Gamma^2(j-k+\beta)\Gamma(j+n)}{\Gamma^2(j-k+1)\Gamma(j+1)}\Bigg)\\
&\approx \sum_{k=1}^{\infty}\sum_{|\alpha|=k}|a_\alpha|^2\omega_\alpha\left\{I_1+I_2\right\},
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= \sum_{j=0}^{\infty}B(j+1,n+\delta)\frac{\Gamma(j+\beta)\Gamma(j+n)}{\Gamma^2(j+1)}\\
&\quad\times\left[\frac{B(j+k+1,n+\delta)\Gamma(j+\beta)\Gamma(j+k+n)}{\Gamma(j+k+1)\Gamma(j+n)}\right.\\
&\qquad\left.-\chi(j-k)\frac{B(j+1,n+\delta)\Gamma(j-k+\beta)}{\Gamma(j-k+1)}\right]\\
&= \lim_{m\to\infty}\left[\sum_{j=0}^{m}B(j+1,n+\delta)\frac{B(j+k+1,n+\delta)\Gamma^2(j+\beta)\Gamma(j+k+n)}{\Gamma^2(j+1)\Gamma(j+k+1)}\right.\\
&\qquad\left.-\sum_{j=k}^{m}\frac{B^2(j+1,n+\delta)\Gamma(j+n)\Gamma(j+\beta)\Gamma(j-k+\beta)}{\Gamma^2(j+1)\Gamma(j-k+1)}\right]
\end{aligned}
\tag{10}
$$

and

$$I_2 = \sum_{j=0}^{\infty} B(j+1, n+\delta)\frac{\Gamma(j+n)}{\Gamma(j+1)}\chi(j-k)\frac{\Gamma(j-k+\beta)}{\Gamma(j-k+1)} \times \left[B(j-k+1, n+\delta)\frac{\Gamma(j-k+\beta)}{\Gamma(j-k+1)} - B(j+1, n+\delta)\frac{\Gamma(j+\beta)}{\Gamma(j+1)}\right]. \tag{11}$$

As in [2] we estimate I_1. To this end we write

$$\begin{aligned}
I_1 &= \lim_{m\to\infty}\left[\sum_{j=0}^{m-k} B(j+1, n+\delta)\frac{B(j+k+1, n+\delta)\Gamma^2(j+\beta)\Gamma(j+k+n)}{\Gamma^2(j+1)\Gamma(j+k+1)}\right. \\
&\quad + \sum_{j=m-k+1}^{m} B(j+1, n+\delta)\frac{B(j+k+1, n+\delta)\Gamma^2(j+\beta)\Gamma(j+k+n)}{\Gamma^2(j+1)\Gamma(j+k+1)} \\
&\quad \left. - \sum_{j=k}^{m}\frac{\Gamma(j+n)\Gamma(j+\beta)B^2(j+1, n+\delta)\Gamma(j-k+\beta)}{\Gamma^2(j+1)\Gamma(j-k+1)}\right] \\
&= \lim_{m\to\infty}\left[\sum_{j=0}^{m-k} B(j+1, n+\delta)\frac{B(j+k+1, n+\delta)\Gamma^2(j+\beta)\Gamma(j+k+n)}{\Gamma^2(j+1)\Gamma(j+k+1)\Gamma(j+n)}\right. \\
&\quad - \sum_{j=0}^{m-k}\frac{B^2(j+k+1, n+\delta)\Gamma(j+k+n)\Gamma(j+k+\beta)\Gamma(j+\beta)}{\Gamma^2(j+k+1)\Gamma(j+1)} \\
&\quad \left. + \sum_{j=m-k+1}^{m} B(j+1, n+\delta)\frac{B(j+k+1, n+\delta)\Gamma^2(j+\beta)\Gamma(j+k+n)}{\Gamma^2(j+1)\Gamma(j+k+1)}\right] \\
&= \lim_{m\to\infty}\left[\sum_{j=0}^{m-k}\frac{B(j+k+1, n+\delta)\Gamma(j+\beta)\Gamma(j+k+n)}{\Gamma(j+1)\Gamma(j+k+1)}\right. \\
&\quad \times\left(B(j+1, n+\delta)\frac{\Gamma(j+\beta)}{\Gamma(j+1)} - B(j+k+1, n+\delta)\frac{\Gamma(j+k+\beta)}{\Gamma(j+k+1)}\right) \\
&\quad \left. + \sum_{j=m-k+1}^{m} B(j+1, n+\delta)\frac{B(j+k+1, n+\delta)\Gamma^2(j+\beta)\Gamma(j+k+n)}{\Gamma(j+k+1)\Gamma^2(j+1)}\right] \\
&= \lim_{m\to\infty}\left[I_{11} + I_{12}\right].
\end{aligned}$$

For $p < 1$, by Lemma C, we have

$$\begin{aligned}
I_{12} &= \lim_{m\to\infty} \sum_{j=m-k+1}^{m} B(j+1, n+\delta)B(j+k+1, n+\delta)\frac{\Gamma^2(j+\beta)\Gamma(j+k+n)}{\Gamma(j+k+1)\Gamma^2(j+1)} \\
&\approx \lim_{m\to\infty} \sum_{j=m-k+1}^{m} (j+k+1)^{-\delta-1}(j+1)^{2\beta-n-\delta-2} \\
&= \lim_{m\to\infty} \sum_{j=m-k+1}^{m} (j+k+1)^{-\delta-1}(j+1)^{p+\delta} \\
&\approx \lim_{m\to\infty} \sum_{j=m-k+1}^{m} j^{p-1} = 0.
\end{aligned}$$

By Lemmas C and D, we have

$$\begin{aligned}
I_{11} &= \lim_{m\to\infty} \sum_{j=0}^{m-k} \frac{B(j+k+1, n+\delta)\Gamma(j+\beta)\Gamma(j+k+n)}{\Gamma(j+1)\Gamma(j+k+1)} \\
&\qquad \times \left(B(j+1, n+\delta)\frac{\Gamma(j+\beta)}{\Gamma(j+1)} - B(j+k+1, n+\delta)\frac{\Gamma(j+k+\beta)}{\Gamma(j+k+1)} \right) \\
&\approx \lim_{m\to\infty} \sum_{j=0}^{m-k} (j+k+1)^{-\delta-1}(j+1)^{\beta-1} \\
&\qquad \times \left(B\left(j+\beta, \frac{n-p}{2}\right) - B\left(j+k+\beta, \frac{n-p}{2}\right)\right) \\
&\approx \lim_{m\to\infty} \sum_{j=0}^{m-k} (j+k+1)^{-\delta-1}(j+1)^{\beta-1} \left((j+1)^{\frac{p-n}{2}} - (j+k+1)^{\frac{p-n}{2}} \right) \\
&\approx \int_1^\infty (x+k)^{-\delta-1}x^{\beta-1}\left(x^{\frac{p-n}{2}} - (x+k)^{\frac{p-n}{2}} \right) dx \\
&\approx k^p \int_{1/k}^\infty \left((t+1)^{-\delta-1}t^{p+\delta} - (t+1)^{\frac{p-n}{2}-\delta-1}t^{\frac{p+n}{2}+\delta} \right) dt \\
&\approx k^p \int_0^\infty \left((t+1)^{-\delta-1}t^{p+\delta} - (t+1)^{\frac{p-n}{2}-\delta-1}t^{\frac{p+n}{2}+\delta} \right) dt \\
&\approx k^p \int_0^\infty t^{p+\delta}(t+1)^{\frac{p-n}{2}-\delta-1} \left((t+1)^{\frac{n-p}{2}} - t^{\frac{n-p}{2}} \right) dt \approx k^p \approx (k+n)^p.
\end{aligned}$$

Thus from the estimates for I_{11} and I_{12}, we have

$$I_1 \approx k^p \approx (k+n)^p.$$

The same is true for I_2, because

$$\begin{aligned}
I_2 &= \lim_{m\to\infty}\sum_{j=k}^{m} B(j+1,n+\delta)\frac{\Gamma(j+n)}{\Gamma(j+1)}\frac{\Gamma(j-k+\beta)}{\Gamma(j-k+1)}\\
&\quad\times\left[B(j-k+1,n+\delta)\frac{\Gamma(j-k+\beta)}{\Gamma(j-k+1)} - B(j+1,n+\delta)\frac{\Gamma(j+\beta)}{\Gamma(j+1)}\right]\\
&= \lim_{m\to\infty}\sum_{j=0}^{m-k} B(j+k+1,n+\delta)\frac{\Gamma(j+k+n)}{\Gamma(j+k+1)}\frac{\Gamma(j+\beta)}{\Gamma(j+1)} \qquad (12)\\
&\quad\times\left[B(j+1,n+\delta)\frac{\Gamma(j+\beta)}{\Gamma(j+1)} - B(j+k+1,n+\delta)\frac{\Gamma(j+k+\beta)}{\Gamma(j+k+1)}\right]\\
&\approx \lim_{m\to\infty}\sum_{j=0}^{m-k}(j+k+1)^{-\delta-1}(j+1)^{\beta-1}\Big((j+1)^{\frac{p-n}{2}} - (j+k+1)^{\frac{p-n}{2}}\Big)\\
&\approx k^p \approx (k+n)^p.
\end{aligned}$$

Thus for $p<1$ and $p+\delta>-1$,

$$I \approx \sum_{k=1}^{\infty}\sum_{|\alpha|=k}(n+|\alpha|)^p|a_\alpha|^2\omega_\alpha,$$

which completes the proof. □

Proof of Theorem 2. Let $n>1$ and $f(z)=\sum_{|\alpha|\ge 0} a_\alpha z^\alpha$ has at least one nonzero coefficient a_α, $|\alpha|=k>0$. As in the proof of Theorem 1, it is enough to consider the case when $\delta=\tau$. We will also use the notation introduced in the proof of Theorem 1. First observe that both series I_1 and I_2, defined by (10) and (11), respectively, have positive terms. Indeed,

$$\begin{aligned}
I_1 &= \sum_{j=0}^{k-1} B(j+1,n+\delta)\frac{B(j+k+1,n+\delta)\Gamma^2(j+\beta)\Gamma(j+k+n)}{\Gamma^2(j+1)\Gamma(j+k+1)}\\
&+\sum_{j=k}^{\infty} B(j+1,n+\delta)\frac{\Gamma(j+\beta)\Gamma(j+n)}{\Gamma^2(j+1)}\\
&\times\left[\frac{B(j+k+1,n+\delta)\Gamma(j+\beta)\Gamma(j+k+n)}{\Gamma(j+k+1)\Gamma(j+n)} - \frac{B(j+1,n+\delta)\Gamma(j-k+\beta)}{\Gamma(j-k+1)}\right]\\
&\approx \sum_{j=0}^{\infty} B(j+k+1,n+\delta)\frac{\Gamma(j+k+\beta)\Gamma(j+k+n)}{\Gamma^2(j+k+1)}\\
&\qquad\times\left[\frac{B(j+2k+1,n+\delta)\Gamma(j+k+\beta)\Gamma(j+2k+n)}{\Gamma(j+2k+1)\Gamma(j+k+n)}\right.\\
&\qquad\qquad\left. - \frac{B(j+k+1,n+\delta)\Gamma(j+\beta)}{\Gamma(j+1)}\right],
\end{aligned}$$

and

$$\frac{B(j+2k+1,n+\delta)\Gamma(j+k+\beta)\Gamma(j+2k+n)}{\Gamma(j+2k+1)\Gamma(j+k+n)} - \frac{B(j+k+1,n+\delta)\Gamma(j+\beta)}{\Gamma(j+1)}$$

$$= \frac{B(j+k+1,n+\delta)\Gamma(j+k+\beta)\prod_{s=1}^{k}(j+k+n-1+s)}{\Gamma(j+k+1)\prod_{s=1}^{k}(j+k+n+\delta+s)} - \frac{B(j+k+1,n+\delta)\Gamma(j+\beta)}{\Gamma(j+1)}$$

$$= \frac{B(j+k+1,n+\delta)\Gamma(j+\beta)}{\Gamma(j+1)}\left[\prod_{s=0}^{k-1}\frac{(j+k+n+s)(j+\beta+s)}{(j+k+n+\delta+1+s)(j+1+s)} - 1\right] > 0,$$

because, putting $\beta = \frac{n+3}{2}+\delta$, we see that $\frac{(j+k+n+s)(j+\frac{n+3}{2}+\delta+s)}{(j+k+n+\delta+1+s)(j+1+s)} > 1$ for $s = 0,1,\ldots,k-1$. For the series I_2, by formula (11), we obtain

$$I_2 = \sum_{j=0}^{\infty} B(j+1,n+\delta)\frac{\Gamma(j+n)}{\Gamma(j+1)}\frac{\Gamma(j-k+\beta)}{\Gamma(j-k+1)} \times \left[B(j+1,n+\delta)\frac{\Gamma(j+\beta)}{\Gamma(j+1)} - B(j+k+1,n+\delta)\frac{\Gamma(j+k+\beta)}{\Gamma(j+k+1)}\right]$$

$$= \sum_{j=0}^{\infty} B^2(j+1,n+\delta)\frac{\Gamma(j+n)\Gamma(j-k+\beta)\Gamma(j+\beta)}{\Gamma^2(j+1)\Gamma(j-k+1)} \times \left[1 - \prod_{s=1}^{k}\frac{(j+\beta-1+s)}{(j+n+\delta+s)}\right]$$

and $\frac{(j+\beta-1+s)}{(j+n+\delta+s)} < 1$ for $s = 0,1,\ldots,k-1$ and $\beta = \frac{n+3}{2}+\delta$.

Now (12) implies

$$I > \sum_{|\alpha|=k} |a_\alpha|^2 \omega_\alpha \{I_1 + I_2\} > \sum_{|\alpha|=k} |a_\alpha|^2 \omega_\alpha I_2$$

$$\approx \sum_{|\alpha|=k} |a_\alpha|^2 \omega_\alpha \sum_{j=0}^{\infty} (j+k+1)^{-\delta-1}(j+1)^{\frac{n+1}{2}+\delta} \times \left[(j+1)^{\frac{1-n}{2}} - (j+k+1)^{\frac{1-n}{2}}\right].$$

Since

$$\lim_{t\to\infty} \frac{(t+k+1)^{-\delta-1}(t+1)^{\frac{n+1}{2}+\delta}\left((t+1)^{\frac{1-n}{2}}-(t+k+1)^{\frac{1-n}{2}}\right)}{(t+k+1)^{-1}}$$

$$= \lim_{t\to\infty} \frac{(1-\frac{k}{t+k+1})^{\frac{n+1}{2}+\delta}\left((1-\frac{k}{t+k+1})^{\frac{1-n}{2}}-1\right)}{(t+k+1)^{-1}} = \frac{k(n-1)}{2},$$

the last series diverges for a given positive integer k. □

Finally, we remark that in view of Theorems 1 and 2 and the results in [4], in the case when $n > 1$, Theorem A is valid if and only if $p < 1$.

It can be also shown by direct calculation that in the case when $p = 1$ and $n > 1$ integral (1) diverges for $f_1(z) = z_1$. Indeed, by (2) we have

$$K = \int_B Q^2_{f_1}(z)(1-|z|^2)^{-2}dv(z) \approx \int_B \frac{1-|z_1|^2}{1-|z|^2}dv(z).$$

Integrating in polar coordinates and using Lemma 1.9 in K. Zhu [7] for $k = 1$ and $n > 1$ (see also W. Rudin [6], p. 14), we get

$$\begin{aligned} K &\approx \int_0^1 r^{2n-1}dr \int_S \frac{1-r^2|\zeta_1|^2}{1-r^2}d\sigma(\zeta) \\ &\approx \int_0^1 \frac{r^{2n-1}}{1-r^2}dr \int_{B_1}(1-r^2|w|^2)(1-|w|^2)^{n-1}dv(w) \\ &\approx \int_0^1 \frac{r^{2n-1}}{1-r^2}\int_0^{2\pi}\int_0^1 t(1-t^2)^{n-1}(1-r^2t^2)dt\,d\varphi\,dr \\ &\approx \int_0^1 \frac{r^{2n-1}}{1-r^2}\int_0^1 (1-y)^{n-1}(1-r^2y)dy\,dr \\ &\approx \int_0^1 \frac{r^{2n-1}}{1-r^2}(n+1-r^2)dr \end{aligned}$$

and the last integral diverges.

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