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ZBIGNIEW A. ŁAGODOWSKI

On the necessary condition for Baum–Katz type theorem for non-identically distributed and negatively dependent random fields

Abstract. Let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ be a random field of negatively dependent random variables. The complete convergence results for negatively dependent random fields are refined. To obtain the main theorem several lemmas for convergence of families indexed by $\mathbb{N}^d$ have been proved. Auxiliary lemmas have wider application to study the random walks on the lattice.

1. Introduction. The concept of complete convergence was introduced in [6] by Hsu and Robbins. They proved that the sequence of arithmetic means of independent, identically distributed (i.i.d.) random variables converges completely to the expected value of the variables, provided the random variables are square-integrable. The result was later generalized to the now classical theorem by Baum and Katz [1] and in this shape was extended to the multidimensional case by Gut in [5]. Therein the normalization is the product of all indices having the same power. Klesov in [7] discussed more general approach to the strong law of large numbers and Baum–Katz type theorems viz., considered normalizing family in general form $b_{\mathbf{n}} = b(n_1, \dots, n_d)$, as a corollary he obtained the case where different indices have different powers in the normalization. This case is also discussed by Gut and Stadtmüller

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in [5, 4], where the authors completed the results of Klesov, giving a consistent approach. General context of these problems is given in recently published monograph by Klesov [8], Chapter 13.5. All above mentioned authors considered fields of independent, identically distributed random variables. Applying a new type of the Kahane–Hoffmann–Jørgensen inequality, Łagodowski [9] proved a few Hsu–Robbins–Erdös–Spitzer–Baum–Katz type theorems for random fields with dependence structures and non-identically distributed. However, the necessary condition in Theorem 3.1 of [9] is not a complete extension of independent identically distributed random fields case. We are going to refine Theorem 3.1 of [9]. To achieve this goal, we proved several results for the convergence of families $\{a_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ and their partial sums. These auxiliary lemmas can be also very useful to study the complete convergence in more general context. Sufficient and necessary conditions for convergence of $\{a_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ as $\max_{1\le i\le d} n_i \to \infty$ in a metric space, allow us to get straightforward extension of results for the metrizable convergence of random variables to analogous results for convergence of random fields.

We will consider random variables on a probability space $(\Omega, \mathfrak{F}, P)$, indexed by the lattice points, i.e. by the index set $\mathbb{N}^d$, $d \ge 2$. The elements of $\mathbb{N}^d$ denoted: $\mathbf{m} = (m_1, m_2, \dots, m_d)$, $\mathbf{n} = (n_1, n_2, \dots, n_d)$, $\mathbf{n}(d) = (n, n, \dots, n)$ etc. are partially ordered by coordinatewise relation:

$$\mathbf{m} \le \mathbf{n} \quad \text{if and only if} \quad m_i \le n_i, \quad i = 1, 2 \dots, d.$$

Throughout the paper, the monotonicity of subfamilies or families is considered with respect to componentwise partial order; only in one case (cf. Lemma 2.4), we use monotonicity with respect to $|\mathbf{n}|$.

A family of random variables $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ we also call a random field and furthermore we denote $S_{\mathbf{n}} = \sum_{\mathbf{k}\le\mathbf{n}} X_{\mathbf{k}}$.

Some additional notation. Let $D = \{1, 2, \dots, d\}$ and J be subset of D, $J^c = D \setminus J$. On $\mathbb{N}^d$ we introduce a relation and functions restricted to subsets of D, i.e. for a given $J \subseteq D$ and $\mathbf{m}, \mathbf{n} \in \mathbb{N}^d$ we define

$$\mathbf{m}\mathcal{R}_J\mathbf{n} \quad \text{if and only if} \quad m_i\mathcal{R}n_i, \quad \text{for} \quad i \in J.$$

Similarly, for a given $\mathbf{n}$ we put $\min_J \mathbf{n} := \min_{i\in J} n_i$ (analogously $\max_J \mathbf{n}$). If $J = D$, we skip the "index" J and write $\mathbf{m}\mathcal{R}\mathbf{n}$, $\max \mathbf{n}$, etc., if $J = \emptyset$, then the relation $\mathcal{R}_J$ is total and the functions are identities.

Recall that $\lim\limits_{\min \mathbf{n}\to\infty} a_{\mathbf{n}} = a$ means

$$\forall_{\varepsilon>0}\, \exists_{\mathbf{k}_0\in\mathbb{N}^d}\, \forall_{\mathbf{n}>\mathbf{k}_0} \quad d(a_{\mathbf{n}}, a) < \varepsilon \tag{1}$$

and $\lim\limits_{\max \mathbf{n}\to\infty} a_{\mathbf{n}} = a$ (sometimes called "strong convergence") means

$$\forall_{\varepsilon>0}\, \exists_{k_0\in\mathbb{N}}\, \forall_{\mathbf{n}:|\mathbf{n}|>k_0} \quad d(a_{\mathbf{n}}, a) < \varepsilon. \tag{2}$$

Let us observe that in (1) we can substitute $\mathbf{k}_0$ by $\mathbf{k_0}(d)$.

2. Auxiliary lemmas. In this section we collect some useful lemmas, maybe some of them are known, but we could not locate a reference. Undermentioned result is the multidimensional version of the theorem for subsequences in a metric space.

Lemma 2.1. *If there exists an element $a \in Y$, such that from every subsequence of family $\{a_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ of elements of metric space (Y, d) we can choose a subsequence that converges to a, then*

$$\lim_{\max \mathbf{n} \to \infty} a_{\mathbf{n}} = a.$$

Proof. The idea of the proof is similar to the one-dimensional case. □

The following lemma gives the sufficient and necessary conditions for the "strong convergence" of $\{a_{\mathbf{n}},\ \mathbf{n} \in \mathbb{N}^d\}$.

Lemma 2.2. *Let $\{a_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ be a family of elements of a metric space (Y, d), thus the following conditions are equivalent:*

(i) $\lim_{\max \mathbf{n} \to \infty} a_{\mathbf{n}} = a$,

(ii) $\forall_{\{\mathbf{n}_l\} \subseteq \mathbb{N}^d} \quad \Big(\{\mathbf{n}_l\}$ *increasing* $\Rightarrow \lim_{l \to \infty} a_{\mathbf{n}_l} = a\Big)$.

Proof. (i) ⇒ (ii)
If $\{\mathbf{n}_l\}$ is increasing, then $\max \mathbf{n}_l \to \infty$ as $l \to \infty$, thus for every $k \in \mathbb{N}$ almost all elements of sequence $\{\mathbf{n}_l\}$ are included in $\{\mathbf{n} : |\mathbf{n}| \geq k\}$.
(i) ⇐ (ii)
Let $\{\mathbf{n}_l\}$ be an infinite subsequence of $\mathbf{n} \in \mathbb{N}^d$, then $|\mathbf{n}_l| \to \infty$ as $l \to \infty$. Thus there exists $\emptyset \neq J \subseteq D$ such that

$$\min_J \mathbf{n}_l \to \infty \quad \text{as} \quad l \to \infty \quad \text{and} \quad \exists_{\mathbf{m} \in \mathbb{N}^d} \forall_{l \in \mathbb{N}}\ \mathbf{n}_l \leq_{J^c} \mathbf{m}.$$

From the subsequence $\{\mathbf{n}_l\}$ we can choose an increasing subsequence $\{\mathbf{n}_{l_k}\}$, which upon the assumption is convergent to a. Now, application of Lemma 2.1 completes the proof. □

Remark 1. Lemma 2.2 means that we can straightforwardly transfer metrizable convergence results for random variables to convergence of random fields as $\max \mathbf{n} \to \infty$. This tool is also very helpful to study the convergence of partial sums, as we will see in the proof of Lemma 2.4. Obviously, convergence in the sense $\max \mathbf{n} \to \infty$ implies convergence in the sense $\min \mathbf{n} \to \infty$, thus we have another sufficient condition without completeness of Y as it was assumed in Lemma V-1-1 of [11].

Lemma 2.3. *Let $\{a_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ be a family of non-negative real numbers and $S_{\mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} a_{\mathbf{k}}$, then the following conditions are equivalent:*

(i) $\lim_{\min \mathbf{n} \to \infty} S_{\mathbf{n}} = S$,

(ii) $\forall_{\{\mathbf{n}_k\}\subseteq\mathbb{N}^d}\ \Big(\{\mathbf{n}_k\}\textit{is increasing}\quad\Rightarrow \lim_{k,l\to\infty}|S_{\mathbf{n}_k}-S_{\mathbf{n}_l}|=0\Big).$

Proof. (i) $\Rightarrow$ (ii) is obvious. To prove (ii) $\Rightarrow$ (i), it is enough to set $\mathbf{n}_k = \mathbf{k}(d)$. □

Lemma 2.4. *Let* $\{a_{\mathbf{i},\mathbf{n}},\ \mathbf{i}\le\mathbf{n}\in\mathbb{N}^d\}$ *be a family of real numbers, non-decreasing with respect to* $|\mathbf{n}|$ *, such that* $0<a_{\mathbf{i},\mathbf{n}}\le 1$ *for* $\mathbf{i}\le\mathbf{n}$ *and* $1-a_{\mathbf{n},\mathbf{n}} = o(\frac{1}{|\mathbf{n}|})$, *then*

$$\sum_{\mathbf{n}\in\mathbb{N}^d}\frac{1}{|\mathbf{n}|}\Big(1-\prod_{\mathbf{i}\le\mathbf{n}}a_{\mathbf{i},\mathbf{n}}\Big)<\infty \quad \textit{implies} \quad \lim_{\max\mathbf{n}\to\infty}\prod_{\mathbf{i}\le\mathbf{n}}a_{\mathbf{i},\mathbf{n}}=1. \tag{3}$$

Proof. By Lemma 2.2 the assertion is equivalent to convergence for any increasing subsequence and let us observe that $\prod_{\mathbf{i}\le\mathbf{n}}a_{\mathbf{i},\mathbf{n}}$ are bounded by 1. On the contrary, assume that there exists increasing sequence $\{\mathbf{n}_k\}$ such that

$$\lim_{k\to\infty}\prod_{\mathbf{i}\le\mathbf{n}_k}a_{\mathbf{i},\mathbf{n}_k}=g<1. \tag{4}$$

Hence, there exist $0<q<1$ and k_0, such that for all $k>k_0$

$$\prod_{\mathbf{i}\le\mathbf{n}_k}a_{\mathbf{i},\mathbf{n}_k}<q. \tag{5}$$

Note, that for $\mathbf{n}\le\mathbf{n}_k$

$$\prod_{\mathbf{i}\le\mathbf{n}}a_{\mathbf{i},\mathbf{n}}\le\prod_{\mathbf{i}\le\mathbf{n}}a_{\mathbf{i},\mathbf{n}_k}=\prod_{\mathbf{i}\le\mathbf{n}_k}a_{\mathbf{i},\mathbf{n}_k}\frac{1}{\prod_{\mathbf{i}\le\mathbf{n}_k,\,\mathbf{i}\not\le\mathbf{n}}a_{\mathbf{i},\mathbf{n}_k}}. \tag{6}$$

Let $\{\mathbf{n}_k\}$ be the sequence as in proof of Lemma 2.2 and define sequence $\{\mathbf{m}_k\}$ as follows

$$\mathbf{m}_k =_{J^c} \mathbf{n}_k \quad\text{and}\quad \mathbf{m}_k =_J \left[\frac{\mathbf{n}_k}{2}\right],$$

where $\alpha\mathbf{n}=(\alpha n_1,\alpha n_2,\dots,\alpha n_d)$.

On the virtue of the assumptions and standard arguments

$$1\ge\prod_{\mathbf{i}\le\mathbf{n}_k,\,\mathbf{i}\not\le\mathbf{m}_k}a_{\mathbf{i},\mathbf{n}_k}\ge\prod_{\mathbf{i}\le\mathbf{n}_k,\,\mathbf{i}\not\le\mathbf{m}_k}a_{\mathbf{i},\mathbf{i}}\to 1\quad\text{as}\quad k\to\infty.$$

Thus, for $0<\varepsilon<1-q$, there exists k_1 such that

$$\forall_{k>k_1}\quad\prod_{\mathbf{i}\le\mathbf{n}_k,\,\mathbf{i}\not\le\mathbf{m}_k}a_{\mathbf{i},\mathbf{n}_k}>1-\varepsilon. \tag{7}$$

On the other hand,

$$\prod_{\mathbf{i}\le\mathbf{n}_k,\,\mathbf{i}\not\le\mathbf{n}}a_{\mathbf{i},\mathbf{n}_k}\ge\prod_{\mathbf{i}\le\mathbf{n}_k,\,\mathbf{i}\not\le\mathbf{m}_k}a_{\mathbf{i},\mathbf{n}_k}\quad\text{for}\quad\mathbf{n}\le\mathbf{n}_k,\ \mathbf{n}>\mathbf{m}_k. \tag{8}$$

In order to finish the proof, let us observe that by (5)–(8), for every k large enough

$$\sum_{\mathbf{n}\not\leq\mathbf{m}_k,\,\mathbf{n}\leq\mathbf{n}_k}\frac{1}{|\mathbf{n}|}\left(1-\prod_{\mathbf{i}\leq\mathbf{n}}a_{\mathbf{i},\mathbf{n}}\right)\geq\sum_{\mathbf{n}>\mathbf{m}_k,\,\mathbf{n}\leq\mathbf{n}_k}\frac{1}{|\mathbf{n}|}\left(1-\frac{1}{1-\varepsilon}\prod_{\mathbf{i}\leq\mathbf{n}_k}a_{\mathbf{i},\mathbf{n}_k}\right)$$
$$\geq\sum_{\mathbf{n}>\mathbf{m}_k,\,\mathbf{n}\leq\mathbf{n}_k}\frac{1}{|\mathbf{n}|}\left(1-\frac{q}{1-\varepsilon}\right)\geq\left(\frac{1}{2}\right)^{|J|}\left(1-\frac{q}{1-\varepsilon}\right)=\delta>0,$$

which is in contradiction with the assumption together with Lemma 2.3. □

Remark 2. Assumption in Lemma 2.4 on asymptotic of $(1-a_{\mathbf{n},\mathbf{n}})$ is crucial. Even in the case $d=1$ if $\limsup_{n\to\infty}(1-a_{n,n})>0$, one can construct an example that series in (3) is convergent but $\prod_{i\leq n}a_{i,n}\nrightarrow 1$ as $n\to\infty$.

Lemma 2.5. *If* $\prod_{\mathbf{i}\leq\mathbf{n}}a_{\mathbf{i},\mathbf{n}}\to 1$ *as* $\max\mathbf{n}\to\infty$*, then for a given* $0<\delta<1$ *and sufficiently large* $\mathbf{n_0}\in\mathbb{N}^d$ *(i.e.* $|\mathbf{n_0}|$ *is sufficiently large)*

$$\forall_{\mathbf{n}\not\leq\mathbf{n_0}}\quad 1-\prod_{\mathbf{i}\leq\mathbf{n}}a_{\mathbf{i},\mathbf{n}}\geq(1-\delta)\sum_{\mathbf{i}\leq\mathbf{n}}(1-a_{\mathbf{i},\mathbf{n}})\,.$$

Proof. See the proof of Lemma 2.1 [9]. □

3. Main results and proofs. At the beginning of this section, let us recall the notion of negative dependence.

Definition 3.1. A finite family of random variables $\{X_{\mathbf{j}},\mathbf{1}\leq\mathbf{j}\leq\mathbf{n}\}$ is said to be negatively dependent (ND) if

$$P\left[\bigcap_{\mathbf{j}\leq\mathbf{n}}(X_{\mathbf{j}}\leq x_{\mathbf{j}})\right]\leq\prod_{\mathbf{j}\leq\mathbf{n}}P(X_{\mathbf{j}}\leq x_{\mathbf{j}})$$

and

$$P\left[\bigcap_{\mathbf{j}\leq\mathbf{n}}(X_{\mathbf{j}}>x_{\mathbf{j}})\right]\leq\prod_{\mathbf{j}\leq\mathbf{n}}P(X_{\mathbf{j}}>x_{\mathbf{j}})$$

for $x_{\mathbf{i}}\in\mathbb{R}$, $\mathbf{i}\leq\mathbf{n}$.

An infinite family is ND if every finite subfamily is ND. The concept of negative quadrant dependence (NQD – in pairs) was introduced by Lehmann in [10]. Let us observe that negatively associated random variables are ND and ND are NQD. For more recent results and comments see the monograph [2] of Bulinski and Shashkin.

Since we are going to prove results for non-identically distributed random variables, the following conditions allow us to formulate them in a simple form as in i.i.d. case and compare them.

Definition 3.2. Random variables $\{X_{\mathbf{k}}, \mathbf{k} \in \mathbb{N}^d\}$ are weakly mean bounded (WMB) by random variable ξ (possibly defined on a different probability space) if and only if there exist some constants $\kappa_1, \kappa_2 > 0$ such that for all $x > 0$ and $\mathbf{n} \in \mathbb{N}^d$

$$\kappa_2 \cdot P(|\xi| > x) \leq \frac{1}{|\mathbf{n}|} \sum_{\mathbf{k} \leq \mathbf{n}} P(|X_{\mathbf{k}}| > x) \leq \kappa_1 \cdot P(|\xi| > x).$$

To state our main results we need some additional notation. Let $\log^+ x := \max(0, \log x)$ and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_d) \in (\frac{1}{2}, +\infty)^d$. Coordinates α_i are arranged in non-decreasing order, $\alpha_1 r \geq 1$, $p = \max\{k : \alpha_k = \alpha_1\}$, and $\mathbf{n}^{\boldsymbol{\alpha}} = (n_1^{\alpha_1}, \dots, n_d^{\alpha_d})$. Now, we are ready to refine the necessary condition in Theorem 3.1 of [9].

Theorem 3.3. *Let $r > 0$, $\alpha_1 > \frac{1}{2}$ and $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ be a random field of negatively dependent random variables, weak mean bounded by ξ.*

(i) *If $\alpha_1 r \geq 2$ and*

$$\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha_1 r - 2} P\Big(\max_{\mathbf{k} \leq \mathbf{n}} |S_{\mathbf{k}}| > |\mathbf{n}^{\boldsymbol{\alpha}}| \varepsilon\Big) < \infty \quad \textit{for all} \quad \varepsilon > 0, \tag{9}$$

then

$$E|\xi|^r (\log^+ |\xi|)^{p-1} < \infty \ \textit{and if} \ r \geq 1, \ E\xi = 0, \tag{10}$$

(ii) *if $\alpha_1 r \in [1, 2)$ and (9) holds, $P(|X_{\mathbf{n}}| > |\mathbf{n}^{\boldsymbol{\alpha}}|) = o(\frac{1}{|\mathbf{n}|})$, then we obtain (10).*

Proof. The assertion (i) is proved in [9], thus we sketch the proof of (ii). The negative and positive part of ND random variables are still ND, then

$$\begin{aligned} P\Big(\max_{\mathbf{k} \leq \mathbf{n}} |S_{\mathbf{k}}| > |\mathbf{n}^{\boldsymbol{\alpha}}| \varepsilon\Big) &\geq P\Big(\max_{\mathbf{k} \leq \mathbf{n}} |X_{\mathbf{k}}| > 2|\mathbf{n}^{\boldsymbol{\alpha}}| \varepsilon\Big) \\ &\geq P\Big(\max_{\mathbf{k} \leq \mathbf{n}} X_{\mathbf{k}}^+ > 2|\mathbf{n}^{\boldsymbol{\alpha}}| \varepsilon\Big) = 1 - P\Big(\bigcap_{\mathbf{k} \leq \mathbf{n}} [X_{\mathbf{k}}^+ \leq 2|\mathbf{n}^{\boldsymbol{\alpha}}| \varepsilon]\Big) \\ &\geq 1 - \prod_{\mathbf{k} \leq \mathbf{n}} P(X_{\mathbf{k}}^+ \leq 2|\mathbf{n}^{\boldsymbol{\alpha}}| \varepsilon) = 1 - \prod_{\mathbf{k} \leq \mathbf{n}} (1 - P(X_{\mathbf{k}}^+ > 2|\mathbf{n}^{\boldsymbol{\alpha}}| \varepsilon)). \end{aligned} \tag{11}$$

Let $a_{\mathbf{k},\mathbf{n}} = P(X_{\mathbf{k}}^+ \leq 2\varepsilon |\mathbf{n}^{\boldsymbol{\alpha}}|))$, thus Lemma 2.4 implies that

$$\prod_{\mathbf{k} \leq \mathbf{n}} (1 - P(X_{\mathbf{k}}^+ > 2\varepsilon |\mathbf{n}^{\boldsymbol{\alpha}}|)) \to 1 \text{ as } \max \mathbf{n} \to \infty.$$

Analogously, we can get

$$\prod_{\mathbf{k} \leq \mathbf{n}} (1 - P(X_{\mathbf{k}}^- > 2\varepsilon |\mathbf{n}^{\boldsymbol{\alpha}}|)) \to 1 \text{ as } \max \mathbf{n} \to \infty.$$

Now, applying Lemma 2.5 with $a_{\mathbf{k},\mathbf{n}} = P(X^+_{\mathbf{k}} \le \varepsilon|\mathbf{n}^{\boldsymbol{\alpha}}|)$ and $\hat{a}_{\mathbf{k},\mathbf{n}} = P(X^-_{\mathbf{k}} \le \varepsilon|\mathbf{n}^{\boldsymbol{\alpha}}|)$, WMB condition and Lemma 2.2 of [5], for sufficiently large $\min \mathbf{n_0}$, we have

$$\begin{aligned}
&\sum_{\mathbf{n}\nleq\mathbf{n_0}} |\mathbf{n}|^{\alpha_1 r-2} P\Big(\max_{\mathbf{k}\le\mathbf{n}} |S_{\mathbf{k}}| > \varepsilon|\mathbf{n}^{\boldsymbol{\alpha}}|\Big)\\
&\quad \ge C_1 \sum_{\mathbf{n}\nleq\mathbf{n_0}} |\mathbf{n}|^{\alpha_1 r-2} P\Big(\max_{\mathbf{k}\le\mathbf{n}} |X_{\mathbf{k}}| > 2\varepsilon|\mathbf{n}^{\boldsymbol{\alpha}}|\Big)\\
&\quad \ge C_2 \sum_{\mathbf{n}\nleq\mathbf{n_0}} |\mathbf{n}|^{\alpha_1 r-2} \sum_{\mathbf{k}\le\mathbf{n}} P(|X_{\mathbf{k}}| > 2\varepsilon|\mathbf{n}^{\boldsymbol{\alpha}}|)\\
&\quad \ge C_3 \sum_{\mathbf{n}\nleq\mathbf{n_0}} |\mathbf{n}|^{\alpha_1 r-1} P(|\xi| > 2\varepsilon|\mathbf{n}^{\boldsymbol{\alpha}}|)\\
&\quad \ge C_4 E|\xi|^r (\log_+ |\xi|)^{p-1}.
\end{aligned}$$

The second part of the assertion, i.e., that if $r \ge 1$, then $E\xi = 0$, is rather known, confer [5] or [3]. □

If $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ is a random field of negatively dependent, identically distributed random variables, then $P(X^+_{\mathbf{k}} \le \varepsilon|\mathbf{n}^{\boldsymbol{\alpha}}|) = a_{\mathbf{n}}$ and $P(X^-_{\mathbf{k}} \le \varepsilon|\mathbf{n}^{\boldsymbol{\alpha}}|) = \hat{a}_{\mathbf{n}}$, for every $\mathbf{k} \le \mathbf{n}$. In this case Lemma 2.4 can be proved without any additional assumptions and we can obtain a generalization of Theorem 1.3 in [5] to negatively dependent random fields.

Theorem 3.4. *Let $r > 0$, $\alpha_1 > \frac{1}{2}$, $\alpha_1 r \ge 1$ and $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ be a random field of negatively dependent, identically distributed as X random variables. If*

$$E|X|^r (\log_+ |X|)^{d-1} < \infty, \quad \text{and if} \quad r \ge 1,\ EX = 0, \tag{12}$$

then

$$\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha_1 r-2} P(|S_{\mathbf{n}}| > |\mathbf{n}^{\boldsymbol{\alpha}}|\varepsilon) < \infty \quad \text{for all} \quad \varepsilon > 0. \tag{13}$$

Conversely, if

$$\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha_1 r-2} P\Big(\max_{\mathbf{k}\le\mathbf{n}} |S_{\mathbf{k}}| > |\mathbf{n}^{\boldsymbol{\alpha}}|\varepsilon\Big) < \infty \quad \text{for all} \quad \varepsilon > 0, \tag{14}$$

then (12) *holds.*

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Zbigniew A. Łagodowski
Department of Mathematics
Lublin University of Technology
Nadbystrzycka 38D
20-618 Lublin
Poland
e-mail: z.lagodowski@pollub.pl

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