A N N A L E S
U N I V E R S I T A T I S M A R I A E C U R I E - S K Ł O D O W S K A
L U B L I N – P O L O N I A
VOL. LXX SECTIO AAA 2015

PROBLEMS WITH ENERGY OF WAVES DESCRIBED BY KORTEWEG – DE VRIES EQUATION

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ABSTRACT

Different forms of the Korteweg – de Vries equation and their invariants are presented. Different formulas for the energy of the system described by KdV equation are compared to each other for fixed and moving coordinate systems. It is shown that the energy conservation holds only in moving coordinate systems.

1. INTRODUCTION

The Korteweg – de Vries equation (KdV equation for short) is a mathematical model of waves on shallow water surfaces. It was first discovered by Boussinesq in 1877 [1] and then rediscovered by Korteweg and de Vries [2]. Later it was shown that KdV equation appears to be a common approximation for several problems in nonlinear physics in weakly nonlinear, dispersive and long wave limit. It is particularly notable as the prototypical example of an exactly solvable model, that is, a nonlinear partial differential equation whose solutions can be exactly and precisely specified.

In this paper we compare the different forms KdV equations dicussed in the literature and forms of invariants (conservation laws) for those equations. We point out also that energy formula not necessarily gives energy conservation for some of those forms.

The idea to do a systematic classification of energy formulas for different forms of KV equation appeared when we encountered problems with formulation of the energy for the second order KdV equations (sometimes called 'extended' KdV equations) in our recent papers [3, 4].

2. DIFFERENT FORMS OF KDV EQUATION

The geometry of the considered shallow water waves is presented in Fig. 1.

In the shallow water wave problem the fluid is assumed to be inviscid and incompressible and its motion to be irrotational. Therefore a velocity potential ϕ is introduced, which satisfies the Laplace equation inside the fluid and appropriate boundary conditions at the surface and bottom.

It is convenient to study the problem in nondimensional variables. The nondimensional variables are defined as follows

$$\tilde{\eta} = \eta/a, \quad \tilde{\phi} = \phi/(l\frac{a}{h}\sqrt{gh}), \quad \tilde{x} = x/l, \quad \tilde{y} = y/h, \quad \tilde{t} = t/(l/\sqrt{gh}), \tag{1}$$

where a is the wave amplitude, h is the depth of the fluid and l is the average wavelength, see Fig. 1.

In the nondimensional variables the set of hydrodynamic equations for 2-dimensional flow takes the following form [3−8] (henceforth all tildes have been omitted)

$$\beta\phi_{xx} + \phi_{yy} = 0. \tag{2}$$

$$\eta_t + \alpha\phi_x\eta_x - \frac{1}{\beta}\phi_y = 0, \quad \text{for} \quad y = 1 + \alpha\eta \tag{3}$$

$$\phi_t + \frac{1}{2}\alpha\phi_x^2 + \frac{1}{2}\frac{\alpha}{\beta}\phi_y^2 + \eta = 0, \quad \text{for} \quad y = 1 + \alpha\eta \tag{4}$$

$$\phi_y = 0, \quad \text{for} \quad y = 0. \tag{5}$$

Equation (2) is the Laplace equation valid for the whole volume of the fluid. Equations (3) and (4) are so-called kinematic and dynamic boundary conditions at the surface, respectively. Equation (5) represents boundary condition at the bottom. For abbreviation all subscripts denote the partial derivatives with respect to particular variables, i.e. $\phi_t = \frac{\partial\phi}{\partial t}$, $\phi_{xx} = \frac{\partial^2\phi}{\partial x^2}$ and so on.

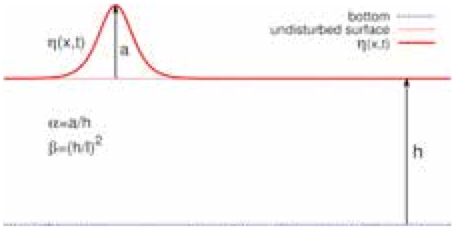

Figure 1: Schematic view of the geometry and definitions of small parameters.

Small parameters α, β are defined by ratios of the wave amplitude a, the average water depth h and mean wavelength l

$$\alpha = \frac{a}{h}, \qquad \beta = \left(\frac{h}{l}\right)^2.$$

KdV equation is obtained from the set (2)-(5) by expansion of the velocity potential in power series with respect to small parameters and neglect of higher order terms. In this way the KdV equation in scaled coordinates in a fixed coordinate system is obtained in the following form

$$\eta_t + \eta_x + \alpha\,\frac{3}{2}\eta\eta_x + \beta\,\frac{1}{6}\eta_{xxx} = 0. \tag{6}$$

Transformation to a *moving frame* in the form

$$\bar{x} = (x - t), \qquad \bar{t} = t, \qquad \bar{\eta} = \eta. \tag{7}$$

allows to remove the term η_x in the KdV equation in a moving frame

$$\bar{\eta}_t + \alpha\,\frac{3}{2}\bar{\eta}\bar{\eta}_{\bar{x}} + \beta\,\frac{1}{6}\bar{\eta}_{\bar{x}\bar{x}\bar{x}} = 0. \tag{8}$$

Problems with mass, momentum and energy conservation in KdV equation were discussed recently in [9]. In the paper the authors considered the KdV equations in the original dimensional variables. Then the KdV equations are

$$\eta_t + c\eta_x + \frac{3}{2}\frac{c}{h}\eta\eta_x + \frac{ch^2}{6}\eta_{xxx} = 0, \quad \text{in a fixed reference frame} \tag{9}$$

$$\eta_t + \frac{3}{2}\frac{c}{h}\eta\eta_x + \frac{ch^2}{6}\eta_{xxx} = 0, \quad \text{in a moving reference frame.} \tag{10}$$

In both, $c = \sqrt{gh}$, and (10) is obtained from (9) by setting $x' = x - ct$ and dropping the prime sign.

Another widely used form of KdV equation is

$$u_t + 6\,u\,u_x + \frac{\beta}{\alpha}u_{xxx} = 0, \qquad \text{or} \qquad u_t + 6\,u\,u_x + u_{xxx} = 0 \quad \text{for} \quad \beta = \alpha. \tag{11}$$

Equations (11), particularly that with $\beta = \alpha$ are favored by mathematicians, see, e.g. [10]. That form of KdV equations is the most convenient for ISM (Inverse Scattering Method) which allows to construct multisoliton solutions, see, e.g. [11-13].

It is worth to note that the equation (11) may be obtained by transformation of (6) to a moving frame with additional scaling, different for space and time variables

$$\bar{x} = \sqrt{\frac{3}{2}}(x - t), \qquad \bar{t} = \frac{1}{4}\sqrt{\frac{3}{2}}\,\alpha\, t, \qquad u = \eta. \tag{12}$$

Therefore we can classify equations (6), (8) and (11) as the KdV equations in scaled nondimensional variables and (9), (10) as KdV equations in dimensional variables. Equations (6) and (9) are referred to as KdV equations in a fixed frame and (8), (10) and (11) as KdV equations in moving frame.

In our present paper we discuss the energy formula obtained for KdV equations both in a fixed frame (6), (9) and a moving frame of reference (8), (10), (11). There seem to be some contradictions in the literature because the form of some invariants and the energy formula differ from each other in different sources because of using different reference frames and/or different scalings. In this paper we aim to show that those different forms are equivalent if one properly 'translates' one form into another.

3. INVARIANTS

In this section we check what kind of invariants can be attributed to equations (6)-(8) and (9)-(10)?

Let T (an analog to density) and X (an analog to flux) are functions which may depend on $x, t, \eta, \eta_x, \eta_{xx}, \ldots$ but not on η_t. If the equation of the form

$$\frac{\partial T}{\partial t} + \frac{\partial X}{\partial x} = 0 \tag{13}$$

holds under some additional conditions then it corresponds to a certain *conservation law*, see, e.g. [14, Ch. 5]. Equation (13) is analogous to continuity equation $\frac{\partial \varrho}{\partial t} + \frac{\partial(\varrho v)}{\partial x} = 0$.

Let functions T and X_x be integrable with respect to x on $(-\infty, \infty)$ and $\lim_{x\to\pm\infty} X$ =const (holds for soliton solutions). Then integration of equation (13) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{-\infty}^{\infty} T\, dx\right) = 0 \qquad \text{or} \qquad \int_{-\infty}^{\infty} T\, dx = \text{const.}\,, \tag{14}$$

because

$$\int_{-\infty}^{\infty} X_x\, dx = X(\infty, t) - X(-\infty, t) = 0 \tag{15}$$

that is conservation law of the quantity $\int_{-\infty}^{\infty} T\, dx = \text{const.}$

The same conclusion holds for periodic solutions (cnoidal waves), when in the integrals (14), (15) limits of integration $(-\infty, \infty)$ are replaced by (a, b), where $b - a = \Lambda$ is the space period of the cnoidal wave (the wave length).

3.1. INVARIANTS OF KDV EQUATION

As it was shown already in [15, 16], for the KdV equation (6) two first invariants can be obtained easily. Writing (6) in the form

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}\left(\eta + \frac{3}{4}\alpha\eta^2 + \frac{1}{6}\beta\eta_{xx}\right) = 0, \tag{16}$$

one obtains immediately the conservarion of mass (volume) law

$$I^{(1)} = \int_{-\infty}^{\infty} \eta\, dx = \text{const}. \tag{17}$$

Similarly, multiplication of (6) by η gives

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\eta^2\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}\eta^2 + \frac{1}{2}\alpha\eta^3 - \frac{1}{12}\beta\eta_x^2 + \frac{1}{6}\beta\eta\eta_{xx}\right) = 0, \tag{18}$$

resulting in the invariant of the form

$$I^{(2)} = \int_{-\infty}^{\infty} \eta^2\, dx = \text{const}. \tag{19}$$

In the literature of the subject, see, e.g. [9, 14], $I^{(2)}$ is attributed to momentum conservation.

Invariants $I^{(1)}, I^{(2)}$ have the same form for all KdV equations (6)-(11).

Let us denote the left hand side of (6) by $\text{KDV}(x,t)$ and take

$$3\eta^2 \times \text{KDV}(x,t) - \frac{2}{3}\frac{\beta}{\alpha}\eta_x \times \frac{\partial}{\partial x}\text{KDV}(x,t). \tag{20}$$

The result, after some simplifications is

$$\frac{\partial}{\partial t}\left(\eta^3 - \frac{1}{3}\frac{\beta}{\alpha}\eta_x^2\right) + \frac{\partial}{\partial x}\left(\frac{9}{8}\alpha\eta^4 + \frac{1}{2}\beta\eta_{xx}\eta^2 - \beta\eta_x^2\eta - \eta^3 + \frac{1}{18}\frac{\beta^2}{\alpha}\eta_{xx}^2 - \frac{1}{9}\frac{\beta^2}{\alpha}\eta_x\eta_{xxx} - \frac{1}{3}\frac{\beta}{\alpha}\eta_x^2\right) - 0. \tag{21}$$

Then the next invariant for KdV equation in fixed reference frame (6) is

$$I^{(3)}_{\text{fixed frame}} = \int_{-\infty}^{\infty}\left(\eta^3 - \frac{1}{3}\frac{\beta}{\alpha}\eta_x^2\right) dx = \text{const}. \tag{22}$$

The same invariant is obtained for the KdV equation in the moving frame (8). The same construction like (20) but for equation (8) results in

$$\frac{\partial}{\partial t}\left(\eta^3 - \frac{1}{3}\frac{\beta}{\alpha}\eta_x^2\right) + \frac{\partial}{\partial x}\left(\frac{9}{8}\alpha\eta^4 + \frac{1}{2}\beta\eta_{xx}\eta^2 - \beta\eta_x^2\eta + \eta^3 + \frac{1}{18}\frac{\beta^2}{\alpha}\eta_{xx}^2 - \frac{1}{9}\frac{\beta^2}{\alpha}\eta_x\eta_{xxx}\right) = 0. \tag{23}$$

Then the next invariant for KdV equation in moving reference frame (6) is

$$I^{(3)}_{\text{moving frame}} = \int_{-\infty}^{\infty}\left(\eta^3 - \frac{1}{3}\frac{\beta}{\alpha}\eta_x^2\right) dx = \text{const}. \tag{24}$$

However, due to different scalings, this invariant looks different for mathematical form of KdV equation in a moving frame (11), that is,

$$I^{(3)}_{\mathrm{math}} = \int_{-\infty}^{\infty} \left(\eta^3 - \frac{1}{2}\eta_x^2 \right) dx = \mathrm{const}. \tag{25}$$

The procedure similar to those described above leads to the same invariants for both equations (9) and (10) where KdV equations are written in dimensional variables. To see this, it is enough to take $3\eta^2 \times kdv(x,t) - \frac{2}{3}h^3 \frac{\partial}{\partial x} kdv(x,t) = 0$, where $kdv(x,t)$ is either (9) or (10). For equation (9) the conservation law is

$$\frac{\partial}{\partial t}\left(\eta^3 - \frac{h^3}{3}\eta_x^2\right) + \frac{\partial}{\partial x}\left(c\eta^3 - \frac{9c}{8h}\eta^4 - \frac{1}{3}ch^3\eta_x^2 - ch^2\eta\eta_x^2 + \frac{1}{2}ch^2\eta^2\eta_{xx} + \frac{1}{18}ch^5\eta_{xx}^2 - \frac{1}{9}ch^5\eta_x\eta_{xxx}\right) = 0. \tag{26}$$

whereas for equation (10) the flux term is different

$$\frac{\partial}{\partial t}\left(\eta^3 - \frac{h^3}{3}\eta_x^2\right) + \frac{\partial}{\partial x}\left(\frac{9c}{8h}\eta^4 - ch^2\eta\eta_x^2 + \frac{1}{2}ch^2\eta^2\eta_{xx} + \frac{1}{18}ch^5\eta_{xx}^2 - \frac{1}{9}ch^5\eta_x\eta_{xxx}\right) = 0. \tag{27}$$

But in both cases the same $I^{(3)}$invariant is obtained as

$$I^{(3)}_{\mathrm{dimensional}} = \int_{-\infty}^{\infty} \left(\eta^3 - \frac{h^3}{3}\eta_x^2 \right) dx = \mathrm{const}. \tag{28}$$

Conclusion. Invariants $I^{(3)}$ have the same form for fixed and moving frames of reference when the transformation from fixed to moving frame scales x and t in the same way (e.g. $x' = x - t$ and $t' = t$). When scaling factors are different, like in (12), then the form of the $I^{(3)}$ in a moving frame differs from the form in a fixed frame.

4. ENERGY

The invariant $I^{(3)}$ is usually refered to as the energy invariant. What is the connection of this invariant with the total energy of the system?

4.1. ENERGY IN THE FIXED COORDINATE SYSTEM CALCULATED FROM THE DEFINITION

Let us construct the total energy of the fluid from the definition.

The total energy is the sum of the potential and the kinetic energy. In our two-dimensional system the energy in original (dimensional coordinates) is

$$E = E_p + E_k = \int_{-\infty}^{+\infty} \left(\int_0^{h+\eta} \rho g y \, dy \right) dx + \int_{-\infty}^{+\infty} \left(\int_0^{h+\eta} \frac{\rho v^2}{2} dy \right) dx. \tag{29}$$

After transformation to scaled nondimensional coordinates (1)

$$E_p = \rho g h^2 l \int_{-\infty}^{+\infty} \int_0^{1+\alpha\tilde{\eta}} \tilde{y} \, d\tilde{y} \, d\tilde{x}. \tag{30}$$

and

$$E_k = \frac{1}{2}\rho g h^2 l \int_{-\infty}^{+\infty} \int_{0}^{1+\alpha\tilde{\eta}} \left(\alpha^2 \tilde{\phi}_{\tilde{x}}^2 + \frac{\alpha^2}{\beta} \tilde{\phi}_{\tilde{y}}^2 \right) d\tilde{y}\, d\tilde{x}. \tag{31}$$

Let us note, that the **factor in front of integrals has the energy dimension**.

In the following, we omit signs $\sim$, having in mind that we are working in dimensionless variables. Integration in (30) with respect to y yields

$$E_p = \frac{1}{2} g h^2 l \rho \int_{-\infty}^{\infty} \left(\alpha^2\eta^2 + 2\alpha\eta + 1\right) dx = \frac{1}{2} g h^2 l \rho \left[\int_{-\infty}^{\infty} \left(\alpha^2\eta^2 + 2\alpha\eta\right) dx + \int_{-\infty}^{\infty} dx \right]. \tag{32}$$

After renormalization (substraction of constant term $\int_{-\infty}^{\infty} dx$) one gets

$$E_p = \frac{1}{2} g h^2 l \rho \int_{-\infty}^{\infty} \left(\alpha^2\eta^2 - 2\alpha\eta\right) dx. \tag{33}$$

In kinetic energy we use velocity potential expressed in the lowest (first) order in small parameters, that is

$$\phi_x = f_x - \frac{1}{2}\beta y^2 f_{xxx} \quad \text{oraz} \quad \phi_y = -\beta y f_{xx}. \tag{34}$$

Then the bracket in the integral (31) is, in the leading orders

$$\left(\alpha^2 {\phi_x}^2 + \frac{\alpha^2}{\beta} {\phi_y}^2 \right) = \alpha^2 \left(f_x^2 + \beta y^2 (-f_x f_{xxx} + f_{xx}^2) \right). \tag{35}$$

Integration with respect to vertical coordinate gives, up to the same order,

$$\begin{aligned} E_k &= \frac{1}{2}\rho g h^2 l \int_{-\infty}^{+\infty} \alpha^2 \left(f_x^2 (1+\alpha\eta) + \beta(-f_x f_{xxx} + f_{xx}^2)\frac{1}{3}(1+\alpha\eta)^3 \right) dx \\ &= \frac{1}{2}\rho g h^2 l \int_{-\infty}^{+\infty} \alpha^2 \left(f_x^2 + \alpha f_x^2 \eta + \frac{1}{3}\beta \left(f_{xx}^2 - f_x f_{xxx} \right) \right) dx. \end{aligned} \tag{36}$$

In order to express energy through the elevation function η we use the relation $f_x = \eta - \frac{1}{4}\alpha\eta^2 + \frac{1}{3}\beta\eta_{xx}$, which is obtained together with KdV equation from the set (2)-(5). Then we substitute $f_x = \eta$ in terms of the third order and $f_x^2 = \eta^2 - \frac{1}{2}\alpha\eta^3 + \frac{2}{3}\beta\eta\eta_{xx}$ in terms of the second order

$$\begin{aligned} E_k &= \frac{1}{2}\rho g h^2 l \int_{-\infty}^{+\infty} \alpha^2 \left[\left(\eta^2 - \frac{1}{2}\alpha\eta^3 + \frac{2}{3}\beta\eta\eta_{xx} \right) + \alpha\eta^3 + \frac{1}{3}\beta \left(\eta_x^2 - \eta\eta_{xx} \right) \right] dx \\ &= \frac{1}{2}\rho g h^2 l\, \alpha^2 \left[\int_{-\infty}^{+\infty} \left(\eta^2 + \frac{1}{2}\alpha\eta^3 \right) dx + \int_{-\infty}^{+\infty} \frac{1}{3}\beta \left(\eta_x^2 + \eta\eta_{xx} \right) dx \right]. \end{aligned} \tag{37}$$

The last term vanishes because from properties of solutions in infinity (15) and integration by parts one obtains

$$\frac{1}{3}\beta \int_{-\infty}^{+\infty} \left(\eta_x^2 + \eta\eta_{xx} \right) dx = \frac{1}{3}\beta \left(\int_{-\infty}^{+\infty} \eta_x^2 dx + \eta\eta_x\Big|_{-\infty}^{\infty} - \int_{-\infty}^{+\infty} \eta_x^2 dx \right) = 0. \tag{38}$$

Therefore the total energy in the fixed frame is given by

$$E_{\rm tot} = E_p + E_k = \frac{1}{2}\rho g h^2 l \int_{-\infty}^{\infty} 2\left(\alpha\eta + (\alpha\eta)^2 + \frac{1}{4}(\alpha\eta)^3\right) dx. \tag{39}$$

The energy (39) in the fixed frame of reference does not contain the $I^{(3)}$ invariant.

The result (39) gives the energy in powers of η only. The same structure of powers in η was obtained by the authors of [9], who work in dimensional KdV equations (9) and (10). On page 122 they present a nondimensional energy density E in a frame moving with the velocity U. Then, if $U = 0$ is set, the energy density in the fixed frame is proportional to $\alpha\eta + \alpha^2\eta^2$ as the formula is obtained up to the second order in α. However, the third order term is $\frac{1}{4}\alpha^3\eta^3$, so the formula up to the third order in α becomes

$$E \sim \alpha\eta + \alpha^2\eta^2 + \frac{1}{4}\alpha^3\eta^3. \tag{40}$$

This energy density contains the same terms as (39) and does not contain the term η_x^2.

Remark. The energy calculated within KdV approximation of hydrodynamic equations in the fixed reference frame **is not expressed by KdV invariants**. In other words this quantity is not necessarily conserved by all solutions of KdV equations. It is conserved only for those solutions which preserve their shapes during motion.

4.2. LUKE'S LAGRANGIAN AND KDV ENERGY

The full set of Euler equations for shallow water problem, as well as KdV equations (6), (11), and higher order KdV equations can be derived from Luke's Lagrangian [17], see, e.g. [6]. Luke pointed out, however, that his Lagrangian is not equal to the difference of kinetic and potential energy. Euler–Lagrange equations obtained from $L = T - V$ have not proper form at the boundary. Instead, Luke's Lagrangian is the sum of kinetic and potential energy suplemented by ϕ_t term which is necessary in dynamical boundary condition.

4.3. LUKE'S LAGRANGIAN IN MARCHANT & SMYTH [6]

The original Lagrangian density in Luke's paper [17] is

$$L = \int_0^{h(x)} \rho\left[\phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2) + gy\right] dy\,. \tag{41}$$

After scaling according to (1) one obtains

$$\phi_t = gh\alpha\,\tilde{\phi}_{\tilde{t}}, \qquad \phi_x^2 = gh\alpha^2\,\tilde{\phi}_{\tilde{x}}^2, \qquad \phi_y^2 = gh\frac{\alpha^2}{\beta}\,\tilde{\phi}_{\tilde{y}}^2. \tag{42}$$

Then the Lagrangian density in scaled variables becomes ($dy = hd\tilde{y}$)

$$L = \rho g h \alpha \int_0^{1+\alpha\eta} \left[\tilde{\phi}_{\tilde{t}} + \frac{1}{2}\left(\alpha\tilde{\phi}_{\tilde{x}}^2 + \frac{\alpha}{\beta}\tilde{\phi}_{\tilde{y}}^2\right)\right] d\tilde{y} + \frac{1}{2}\rho g h^2(1+\alpha\eta)^2. \tag{43}$$

So, in dimensionless quantities

$$\frac{L}{\rho g h a} = \int_0^{1+\alpha\eta} \left[\tilde{\phi}_t + \frac{1}{2}\left(\alpha\tilde{\phi}_{\tilde{x}}^2 + \frac{\alpha}{\beta}\tilde{\phi}_{\tilde{y}}^2\right)\right] d\tilde{y} + \frac{1}{2}\alpha\eta^2. \tag{44}$$

where constant term and term proportional to η in expansion of $(1+\alpha\eta)^2$ are omitted as invariants. The form (44) is identical with the eq. (2.9) in Marchant & Smyth paper [6].

The full Lagrangian is obtained by integration with respect to x, so in dimensionless variables ($dx = l\,d\tilde{x}$) it gives

$$\mathcal{L} = \rho g h a l \int_{-\infty}^{\infty} \left[\int_0^{1+\alpha\eta} \left[\tilde{\phi}_t + \frac{1}{2}\left(\alpha\tilde{\phi}_{\tilde{x}}^2 + \frac{\alpha}{\beta}\tilde{\phi}_{\tilde{y}}^2\right)\right] d\tilde{y} + \frac{1}{2}\alpha\eta^2\right] d\tilde{x}. \tag{45}$$

The factor in front of the integral, $\rho g h a l = \rho g h^2 l\,\alpha$, has the energy dimension.

Next, the sign ($\sim$) will be omitted, but we have to remember that we are working in scaled dimensionless variables in the fixed reference frame.

4.4. ENERGY IN THE FIXED REFERENCE FRAME

Let us express Lagrangian density by η and $f = \phi^{(0)}$ appearing in KdV equation and velocity potential. Then, up to the first order in small parameters

$$\phi = f - \frac{1}{2}\beta y^2 f_{xx}, \qquad \phi_t = f_t - \frac{1}{2}\beta y^2 f_{xxt}, \qquad \phi_x = f_x - \frac{1}{2}\beta y^2 f_{xxx}, \qquad \phi_y = -\beta y f_{xx}. \tag{46}$$

Then the expression under the integral in (44) is

$$[\,] = f_t - \frac{1}{2}\beta y^2 f_{xxt} + \frac{1}{2}\alpha f_x^2 + \frac{1}{2}\alpha\beta y^2\left(-f_x f_{xxx} + f_{xx}^2\right). \tag{47}$$

From properties of solutions at integration limits (over x) $(-f_x f_{xxx} + f_{xx}^2) \Rightarrow 2f_{xx}^2$. Integration (47) over y yields

$$\frac{L}{\rho g h a} = \left(f_t + \frac{1}{2}\alpha f_x^2\right)(1+\alpha\eta) - \frac{1}{2}\beta f_{xxt}\,\frac{1}{3}(1-\alpha\eta)^3 - \alpha\beta f_{xx}^2\,\frac{1}{3}(1+\alpha\eta)^3 + \frac{1}{2}\alpha\eta^2. \tag{48}$$

Then the dimensionless Hamiltonian density is ($f_t\frac{\partial L}{\partial f_t} + f_{xxt}\frac{\partial L}{\partial f_{xxt}} - L$)

$$\frac{H}{\rho g h^2 l} = -\alpha\left[\frac{1}{2}\alpha f_x^2(1+\alpha\eta) + \alpha\beta f_{xx}^2\,\frac{1}{3}(1+\alpha\eta)^3 + \frac{1}{2}\alpha\eta^2\right]. \tag{49}$$

Again, we need to express Hamiltonian by η and its derivatives, only. Inserting into (49)

$$f_x = \eta - \frac{1}{4}\alpha\eta^2 + \frac{1}{3}\beta\eta_{xx} \tag{50}$$

and leaving at most terms of the third order one obtains

$$\frac{H}{\rho g h^2 l} = -\alpha\left[\alpha\eta^2 + \frac{1}{4}\alpha^2\eta^3 + \frac{1}{3}\alpha\beta(\eta_x^2 + \eta\eta_{xx})\right]. \tag{51}$$

So, energy is

$$\frac{E}{\rho g h^2 l} = -\alpha \int_{-\infty}^{\infty} \left[\alpha\eta^2 + \frac{1}{4}\alpha^2\eta^3 + \frac{1}{3}\alpha\beta(\eta_x^2 + \eta\eta_{xx})\right] dx = -\left[\alpha^2 \int_{-\infty}^{\infty} \eta^2 dx + \frac{1}{4}\alpha^3 \int_{-\infty}^{\infty} \eta^3 dx\right] \quad (52)$$

because the integral of the term with $\alpha\beta$ vanishes due to properties of solutions at integration limits. Here, in the same way as in calculations of energy directly from the definition (39), the energy is expressed by integrals of η^2 and η^3. The term proportional to $\alpha\eta$ is not present in (52), because it was dropped earlier.

4.5. ENERGY IN THE MOVING FRAME

Let us do transformation to the moving frame

$$\bar{x} = x - t, \qquad \bar{t} = \alpha t, \qquad \partial_x = \partial_{\bar{x}}, \qquad \partial_t = -\partial_{\bar{x}} + \alpha\partial_{\bar{t}}. \quad (53)$$

Then

$$\phi = f - \frac{1}{2}\beta y^2 f_{xx}, \qquad \phi_x = f_x - \frac{1}{2}\beta y^2 f_{xxx}, \qquad \phi_y = -\beta y f_{xx} \quad (54)$$

and

$$\phi_t = -f_x + \frac{1}{2}\beta y^2 f_{xxx} + \alpha(f_t - \frac{1}{2}\beta y^2 f_{\bar{x}\bar{x}\bar{t}}). \quad (55)$$

Then up to the second order

$$\frac{1}{2}\left(\alpha\phi_x^2 + \frac{\alpha}{\beta}\phi_y^2\right) = \frac{1}{2}\left[\alpha f_x^2 - \alpha\beta y^2(-f_{\bar{x}} f_{\bar{x}\bar{x}\bar{x}} + f_{\bar{x}\bar{x}}^2)\right] = \frac{1}{2}\alpha f_x^2 + \alpha\beta y^2 f_{xx}^2. \quad (56)$$

Therefore the expression under the integral in (44) is

$$[\;] = -f_x + \frac{1}{2}\beta y^2 f_{xxx} + \alpha(f_{\bar{t}} - \frac{1}{2}\beta y^2 f_{\bar{x}\bar{x}\bar{t}}) + \frac{1}{2}\alpha f_{\bar{x}}^2 + \alpha\beta y^2 f_{\bar{x}\bar{x}}^2. \quad (57)$$

Integration yields

$$\frac{L}{\rho g h a} = \left(-f_{\bar{x}} + \alpha f_{\bar{t}} + \frac{1}{2}\alpha f_{\bar{x}}^2\right)(1+\alpha\eta) + \frac{1}{3}(1+\alpha\eta)^3\left(\frac{1}{2}\beta(f_{\bar{x}\bar{x}\bar{x}} - f_{\bar{x}\bar{x}\bar{t}}) + \alpha\beta f_{\bar{x}\bar{x}}^2\right) + \frac{1}{2}\alpha\eta^2. \quad (58)$$

As in (49), the Hamiltonian density is

$$\frac{H}{\rho g h^2 l} = -\alpha\left[\left(-f_x + \frac{1}{2}\alpha f_x^2\right)(1+\alpha\eta) + \frac{1}{3}(1+\alpha\eta)^3\left(\frac{1}{2}\beta f_{xxx} - \alpha\beta f_{xx}^2\right) + \frac{1}{2}\alpha\eta^2\right]. \quad (59)$$

Expressing f_x by (50) one finally obtains

$$\frac{H}{\rho g h^2 l} = -\alpha\left[-\frac{1}{4}\alpha\eta^2 + \frac{1}{3}\beta\eta_{xx} - \frac{1}{2}\alpha^2\eta^3 + \alpha\beta\left(-\frac{1}{4}\eta_x^2 - \frac{5}{12}\eta\eta_{xx}\right) - \frac{1}{18}\beta^2\eta_{xxxx}\right]. \quad (60)$$

Finally the energy is

$$\frac{E}{\rho g h^2 l} = \alpha^2 \frac{1}{4} \int_{-\infty}^{\infty} \eta^2 dx + \alpha^3 \frac{1}{2} \int_{-\infty}^{\infty} \left(\eta^3 - \frac{1}{3} \frac{\beta}{\alpha} \eta_x^2 \right) dx \tag{61}$$

because integrals from terms with β, β^2 vanish at integration limits, and $-\frac{5}{12}\eta\eta_{xx} \Rightarrow \frac{5}{12}\eta_x^2$ by integration by parts. The invariant term proportional to $\alpha\eta$ is not present in (61), because it was dropped already in (44). If we include that term, the total energy becomes a linear combination of all three lowest invariants, $I^{(1)}, I^{(3)}, I^{(3)}$.

Comment. Almost identical formula for the energy in the moving frame for KdV equation, expressed in dimensional variables (10), was obtained in [9]. That energy is expressed also by three lowest order invariants

$$\mathcal{E} = -\frac{1}{2}c^2 \int_{-\infty}^{\infty} \eta \, dx + \frac{1}{4}\frac{c^2}{h} \int_{-\infty}^{\infty} \eta^2 \, dx + \frac{1}{2}\frac{c^2}{h^2} \int_{-\infty}^{\infty} \left(\eta^3 - \frac{h^3}{3}\eta_x^2 \right) dx. \tag{62}$$

5. NUMERICAL EXAMPLES

There are two kinds of solutions to KdV equations. The first ones are the periodic solutions (so-called cnoidal waves, see, e.g. [18, Ch. 13]). The second kind are soliton solutions. Cnoidal solutions as well as single soliton solutions preserve their shapes during motion and therefore all integrals of the forms

$$\int_{-\infty}^{\infty} \eta^a dx, \qquad \int_{-\infty}^{\infty} \eta_x dx, \qquad \ldots \qquad \int_{-\infty}^{\infty} \eta_{nx} dx, \ldots,$$

where $a \in \mathbb{R}$ and $n \in \mathcal{N}$ are arbitrary, are invariants for these solutions. Threfore, for solutions of that kind the energy in forms (39), (40), (52) is conserved. It is not necessarily true for multisoliton solutions. A violation of energy conservation in the fixed frame should be maximal when solitons overlap during scattering. How big is that violation? To obtain some estimations we calculated time evolution of two-soliton solution of KdV equations and energy as function of time for such solution.

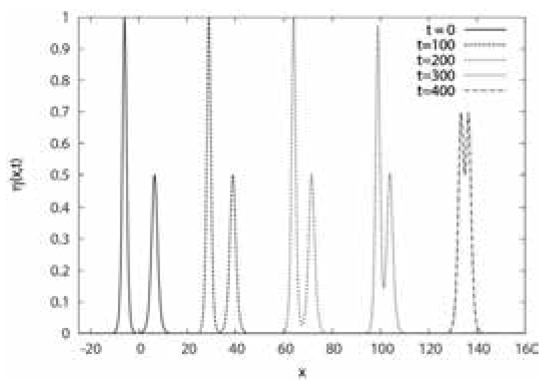

Figure 2: Shapes of two-soliton solution are presented for five instants of time from position of separated solitons to their full overlap. In order to show details of the wave the distances between solitons have been artificially shrinked for $t > 0$.

In Figure 2 shapes of two-soliton solution are presented for five instants of time from position of separated solitons to their full overlap. Calculations were performed in fixed coordinate system.

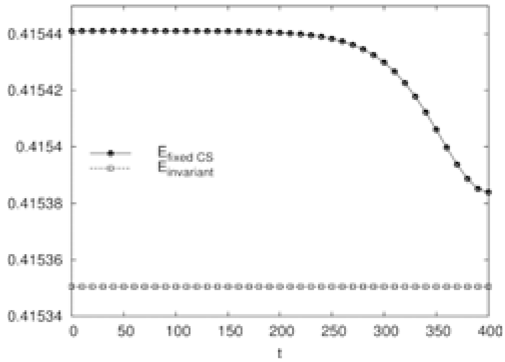

Figure 3: Energy calculated from definition and from invariants of KdV equation. Open dots correspond to results from the formula (63), open squares to those from (64).

In Figure 3 the energy in fixed coordinate system (39) is plotted as a function of time for the motion of two-soliton solution presented in Figure 2 and compared with the corresponding formula expressed by invariants. Precisely, the open dots represent dimensionless part of the total energy in fixed coordinate frame (39), that is

$$E_{\text{fixed CS}} = \int_{-\infty}^{\infty} \left(\alpha\eta + (\alpha\eta)^2 + \frac{1}{4}(\alpha\eta)^3 \right) dx, \tag{63}$$

whereas the open squares show time dependence of

$$E_{\text{invariant}} = \int_{-\infty}^{\infty} \left[\alpha\eta + (\alpha\eta)^2 + \frac{1}{4}\alpha^3 \left(\eta^3 - \frac{1}{3}\frac{\beta}{\alpha}\eta_x^2 \right) \right] dx = \alpha I^{(1)} + \alpha^2 I^{(2)} + \frac{1}{4}\alpha^3 I^{(3)}. \tag{64}$$

The last expression corresponds to the energy in the moving frame (61) in which the term of the first order in α is not dropped. From this figure we see that the relative changes of the energy (39) are of the order of 0.014 %. So, the violation of the energy conservation is indeed very small. The $E_{\text{invariant}}$ as expressed by analytical invariants of KdV equation is constant up to 14 decimal digits in this calculations.

In our numerical calculations of time evolution of solutions to KdV equation we were using finite difference method with periodic boundary conditions. In order to obtain the wave profiles with very good precision we used long space intervals, mesh sizes of several thousands points and appropriate small time steps. For more details we refer to [3, 4].

6. CONCLUSIONS

The main conclusions can be formulated as follows:

- The invariants of KdV equations in fixed and moving frames have the same form only when in the transformation between frames there is the same scaling factor for x and t.
- The usual form of the energy $H = T + V$ is not expressed by invariants only. The reason lies in the fact, as pointed out by Luke, that Euler–Lagrange equations obtained from the Lagrangian $L = T - V$ do not supply right equations at the boundary. However, the energy $H = T + V$ decreases by a very small fraction only for those time instants when solitons composing multisoliton solution overlap.
- In moving coordinate system energy resulting from Luke's Lagrangian density is expressed by invariants of KdV equations.
- Numerical tests confirm that invariants $I^{(1)}, I^{(2)}, I^{(3)}$ in forms (17), (19), (22), (24) are exact constants of motion for two- and three-soliton solutions, both for fixed and moving coordinate systems. In all performed tests the invariants were exact up to thirteen digits in double precision calculations.

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