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Annales UMCS Informatica AI 1 (2003) 49-59

## **Annales UMCS**

Informatica Lublin-Polonia Sectio AI

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### Two hierarchies of $\mathbf{R}$ -recursive functions

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#### Abstract

In the paper some aspects of complexity of  $\mathbf{R}$ -recursive functions are considered. The limit hierarchy of  $\mathbf{R}$ -recursive functions is introduced by the analogy to the  $\mu$ -hierarchy. Then its properties and relations to the  $\mu$ -hierarchy are analysed.

#### 1. Introduction

The classical theory of computation deals with the functions on enumerable (especially natural) domains. The fundamental notion in this field is the notion of a (partial) recursive function. The problem of hierarchies for these functions is also in the interest of mathematicians (for elementary, primitive recursive function, Grzegorczyk hierarchy, compare [1].

During past years many mathematicians have been interested in creating analogous models of computation on real numbers (see for example Grzegorczyk [2], Blum, Shub, Smale [3]). An interesting approach was given by Moore. In the work [4] he defined a set of functions on the reals  $\boldsymbol{R}$  (called  $\boldsymbol{R}$ -recursive functions) in the analogous way to the classical recursive functions on the natural numbers  $\boldsymbol{N}$ . His model has a continuous time of computation (a continuous integration instead of a discrete recursion). The great importance in Moore's model has the zero-finding operation  $\mu$ , which is used to construct  $\mu$ -hierarchy of  $\boldsymbol{R}$ -recursive functions.

It was shown [5] that the zero-finding operator  $\mu$  can be replaced by the operation of infinite limits. This allows us to define a limit hierarchy and relate it to  $\mu$ -hierarchy.

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#### 2. Preliminaries

We start with a fundamental definition of a class of real functions called  $\mathbf{R}$ recursive functions [4].

**Definition 2.1** The set of  $\mathbf{R}$ -recursive functions is generated from the constants 0,1 by the operations:

- 1) composition:  $h(\overline{x}) = f(g(\overline{x}))$ ;
- 2) differential recursion:  $h(\overline{x},0) = f(\overline{x}), \partial_y h(\overline{x},y) = g(\overline{x},y,h(\overline{x},y))$  (the equivalent formulation can be given by integrals:  $h(\overline{x},y) = f(\overline{x}) + \int_0^y g(\overline{x},y',h(\overline{x},y'))dy'$ );
- 3) **m**-recursion  $h(\overline{x}) = \mathbf{m}_y f(\overline{x}, y) = \inf\{y : f(\overline{x}, y) = 0\}$ , where infimum chooses the number  $\mathbf{y}$  with the smallest absolute value and for two  $\mathbf{y}$  with the same absolute value the negative one;
- 4) vector-valued functions can be defined by defining their components.

Several comments are needed to the above definition. A solution of a differential equation need not be unique or can diverge. Hence, we assume that if **h** is defined by a differential recursion then **h** is defined only where a finite and unique solution exists. This is why the set of **R**-recursive functions includes also partial functions. We use (after [4]) the name of **R**-recursive functions in the article, however we should remember that in reality we have partiality here (partial **R**-recursive functions).

The second problem arises with the operation of infimum. Let us observe that if an infinite number of zeros accumulates just above some positive *y* or just below some negative *y* then the infimum operation returns that *y* even if it itself is not a zero.

In the papers [5, 6] it was shown that if in the Moore's definition [4] m operation is replaced by infinite limits:  $h(\overline{x}) = \liminf_{y \to \infty} g(\overline{x}, y)$ ,  $h(\overline{x}) = \limsup_{y \to \infty} g(\overline{x}, y)$  then the resulting class of functions remains the same.

This gives us also the following result (including the limit operation in the form  $h(\overline{x}) = \lim_{y \to \infty} g(\overline{x}, y)$ , which can be in the obvious way obtained from limsup, liminf:

**Corollary 2.2** The class of **R**-recursive functions is closed under the operations of infinite limits:  $h(\overline{x}) = \liminf_{y \to \infty} g(\overline{x}, y)$ ,  $h(\overline{x}) = \limsup_{y \to \infty} g(\overline{x}, y)$ .

#### 3. Hierarchies

The operator m is a key operator in generating the R-recursive functions. In a physical sense it has a property of being strongly uncomputable. This fact suggests creating a hierarchy, which is built with respect to the number of uses of m in the definition of a given f.

**Definition 3.1** ([4]) For a given **R**-recursive expression  $s(\overline{x})$ , let  $M_{x_i}(s)$  (the **m**-number with respect to  $x_i$ ) be defined as follows:

$$M_x(0) = M_x(1) = M_x(-1) = 0,$$
 (1)

$$M_{x}\left(f\left(g_{1},g_{2},...\right)\right) = \max_{i}\left(M_{x_{j}}\left(f\right) + M_{x}\left(g_{j}\right)\right),\tag{2}$$

$$M_{x}\left(h=f+\int_{0}^{y}g(\overline{x},y',h)dy'\right)=\max\left(M_{x}(f),M_{x}(g),M_{h}(g)\right),\tag{3}$$

$$M_{y}\left(h=f+\int_{0}^{y}g\left(\overline{x},y',h\right)dy'\right)=\max\left(M_{y'}\left(g\right),M_{h}\left(g\right)\right),\tag{4}$$

$$M_{x}\left(\boldsymbol{m}_{y}f\left(\overline{x},y\right)\right) = \max\left(M_{x}\left(f\right),M_{y}\left(f\right)\right) + 1,\tag{5}$$

where x can be any  $x_1,...,x_n$  for  $\overline{x} = (x_1,...,x_n)$ 

For an **R**-recursive function f, let  $M(f) = \max_{x_i} (s)$  minimized over all expressions s that define f. Now we are ready to define M-hierarchy (m-hierarchy) as a family of  $M_i = \{f : M'(f) \le j\}$ .

Let us construct the analogous definition of L-hierarchy by replacing in the above definition  $M_x$  by  $L_x$  and changing line (5) to the following form (5'):

$$L_{x}\left(\underset{y\to\infty}{\text{liminf }}g\left(\overline{x},y\right)\right) = L_{x}\left(\underset{y\to\infty}{\text{lim sup }}g\left(\overline{x},y\right)\right) =$$

$$= L_{x}\left(\underset{y\to\infty}{\text{lim }}g\left(\overline{x},y\right)\right) = \max\left(L_{x}\left(f\right),L_{y}\left(f\right)\right) + 1.$$

For an **R**-recursive function f, let  $L(f) = \max_i L_{x_i}(s)$  minimized over all expressions **s** that define **f** without using the **m**-operation.

**Definition 3.2** The **L**-hierarchy is a family of  $L_j = \{f : L(f) \le j\}$ .

Let us add that in Definition 3.2 we use explicitly the operator  $f(\overline{x}) = \lim_{y \to \infty} g(\overline{x}, y)$  to avoid its construction by other operators ( $\limsup$  lim inf), which would effect in a superficially higher class of a complexity of a function f.

As an obvious corollary from definitions we have the following statement.

**Lemma 3.3** The classes  $M_0$  and  $M_1$  are identical.

A function  $f \in L_0 = M_0$  will be called (by an analogy to the case of natural recursive functions) a primitive **R**-recursive function. After Moore [4] we can conclude that such functions as: -x, x+y, xy, x/y,  $e^x$ ,  $\ln x$ ,  $y^x$ ,  $\sin x$ ,  $\cos x$  are primitive **R**-recursive.

We can give a few results on some levels of the limit hierarchy.

**Lemma 3.4.** The Kronecker d function, the signum function and absolute value belong to the first level ( $L_1$ ) of limit hierarchy.

**Proof.** It is sufficient to take the following definitions [5]: hence d(0) = 1 and for all  $x \neq 0$  we have d(x) = 0 let us define  $d(x) = \liminf_{y \to \infty} \left(\frac{1}{1+x^2}\right)^y$ . Now from the expression  $\liminf_{y \to \infty} \arctan xy = \begin{cases} p/2, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -p/2, & \text{if } x < 0, \end{cases}$  sgn $(x) = \frac{\liminf_{y \to \infty} \arctan xy}{2 \arctan 1}$  and  $|x| = \operatorname{sgn}(x)x$ .

We should be careful with definions of functions by cases:

$$\textbf{Lemma 3.5 For } h(\overline{x}) = \begin{cases} g_1(\overline{x}), & \text{if } f(\overline{x}) = 0, \\ g_2(\overline{x}), & \text{if } f(\overline{x}) = 1, \\ \mathbf{M} & \mathbf{M} \\ g_k(\overline{x}), & \text{if } f(\overline{x}) \ge k - 1 \end{cases} \text{ and } g_i \in L_{n_i} \text{ for all } 1 \le i \le k,$$

 $f \in L_m$  the function h belongs to  $L_{\max(n_1,...n_k,m+1)}$ 

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**Proof.** Let us see that 
$$eq(x,y) = d(x-y) \in L_1$$
 and  $ge(x,y) = \frac{\left(\operatorname{sgn}(x-y) + eq(x,y)\right)}{2} + \frac{1}{2} \in L_1$ . Then of course  $h(\overline{x}) = \sum_{i=1}^{k-1} g_i(\overline{x}) eq(f(\overline{x}),i-1) + g_k(\overline{x}) ge(f(\overline{x}),k-1)$ .

Of course this result can be easily extended to other forms of definitions by cases.

**Lemma 3.6** The function  $\Theta(x)$  (equal to 1 if  $x \ge 0$ , otherwise 0), maximum  $\max(x, y)$ , square-wave function  $\mathbf{s}$  are in  $L_2$ , the function p(x) such that p(x)=1 for  $x \in [2n,2n+1]$  and p(x)=0 for  $x \in [2n+1,2n+2]$  is in  $L_2$  and the floor function |x| is in  $L_3$ .

**Proof.** We give the proper definitions (from [6]) for these functions. Let

$$\Theta(x) = d(x - |x|),$$

$$\max(x, y) = xd(x - y) + (1 - d(x - y))[x\Theta(x - y) + y\Theta(y - x)],$$

$$s(x) = \Theta(\sin(px)).$$

The function 
$$p(x)$$
 can be given as  $s(x)\left(1-d\left(\sin\frac{(x-1)p}{2}\right)\right)$ , so  $p \in L_2$ .

The floor function we can define by the auxiliary function w(0) = 0,  $\partial_x w(x) = 2\Theta(-\sin(2px))$  as

$$\lfloor x \rfloor = \begin{cases} 2w(x/2) & \text{if } p(x) = 1, \\ 2w((x-1)/2) & \text{if } p(x) = 0. \end{cases}$$

From the above equation we have [x] in  $L_3$ .

Let us recall that if  $f: \mathbb{R}^n \to \mathbb{R}$  is an **R**-recursive function then the function  $f_{iter}(i, \overline{x})$  is **R**-recursive, too.

**Lemma 3.7** Let  $f: \mathbb{R}^n \to \mathbb{R}$  belongs to the class  $L_i$ , then we have  $f_{iter}: \mathbb{R}^{n+1} \to \mathbb{R}$  is in  $L_{\max(2,j)}$ .

**Proof.** The definitions, which were given by Moore [3]  $f_{iter}(i, \overline{x}) = h(2i)$ , where  $h(0) = g(0) = \overline{x}$ ,

$$\partial_{t}g(t) = \left[f(h(t)) - h(t)\right]s(t),$$

$$\partial_{t}h(t) = \left[\frac{g(t) - h(t)}{r(t)}\right](1 - s(t)),$$

with s - a square wave function in  $L_2$  and r(0) = 0,  $\partial_t r(t) = 2s(t) - 1$ ,  $r, s \in L_2$  give us the desirable statement.  $\square$ 

**Lemma 3.8** The  $\mathbf{R}^l$ -recursive functions  $\mathbf{g}_2: R^2 \to R$ ,  $\mathbf{g}_2^1, \mathbf{g}_2^2: R \to R$  such that  $(\forall x, y \in R) \mathbf{g}_2^1 (\mathbf{g}_2(x, y)) = x$ ,  $(\forall x, y \in R) \mathbf{g}_2^2 (\mathbf{g}_2(x, y)) = y$ , have the following properties:  $\mathbf{g}_2$ ,  $\mathbf{g}_2^1$  are in  $L_{10}$ ,  $\mathbf{g}_2^2$  is in  $L_{14}$ .

**Proof.** We have the auxiliary functions  $\Gamma_2$ ,  $\Gamma_2^1$ ,  $\Gamma_2^2$ , which are coding and decoding functions in the interval (0,1):  $\Gamma_2(x,y) = c(x) + c(y)/10$ , where

$$c(x) = \lim z(a(i,x))/10^{2i} + b(i,x)/10^{i}$$
,

and later  $z(x) = \lim_{i \to \infty} z_{iter}(i, x)$ ,

$$\begin{split} z_{iter}^{\cdot} \left( i, a_{1} \dots a_{n} . a_{n+1} \dots \right) &= a_{1} \dots a_{n} 0 \dots a_{n+1} 0 . a_{n+i+1} \dots, \\ a \left( i, 0 . a_{1} a_{2} \dots a_{i} \dots \right) &= 0 . a_{1} \dots a_{i} \\ b \left( i, 0 . a_{1} a_{2} \dots a_{i} \dots \right) &= 0 . \mathbf{Q}_{1} \dots 0 \ a_{i+1} \dots, \end{split}$$

$$(z'(x)) = \begin{cases} 100 \lfloor x \rfloor + 10(x - \lfloor x \rfloor), & \text{if } \lfloor x \rfloor \neq x, \\ x, & \text{if } \lfloor x \rfloor = x; \end{cases} \in L_4, a, b \in L_4. \text{ Also } z_{iter} \text{ belongs}$$

to  $L_4$ , hence  $\Gamma_2(x,y) \in L_{10}$ , decoding of the first element is described in the symmetric way so  $\Gamma_2^1(x)$  is in  $L_{10}$ , but  $\Gamma_2^2(x) = \Gamma_2^1(10 - \lfloor 10x \rfloor)$  so  $\Gamma_2^2 \in L_{14}$ .

The functions  $\Gamma_2$ ,  $\Gamma_2^1$ ,  $\Gamma_2^2$  can be extended to all reals by one-to-one  $f:(0,1) \to R \in L_0$  without the loss of their class.  $\square$ 

The same method of coding and decoding by interlacing of ciphers (only the power of 10 should be changed) gives us the functions  $g_n: R^n \to R$  and  $g_n^i: R \to R$  for i = 1,...,n such that

$$(\forall i)(\forall x_1...,x_n \in R)g_n^i(g_n(x_1,...,x_n)) = x_i$$

in the same class:  $g_n, g_n^1 \in L_{10}$  and  $(\forall i > 1)g_n^i \in L_{14}$ .

We finish this part with the important form of defining: a new function is given as a product of values f in some integer points.

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**Lemma 3.9** There exists such constant  $p \in N$  that for the function

$$\prod_{z=0}^{y} f(\overline{x}, z) = \begin{cases} f(\overline{x}, 0) f(\overline{x}, 1) ... f(\overline{x}, \lfloor y - 1 \rfloor), & \text{if } y \ge 1, \\ 1, & \text{if } 0 \le y < 1, \\ 0, & \text{if } y < 0, \end{cases}$$

if the function f is in the class  $L_m$  then  $\prod_{z=0}^y f(\overline{x}, z)$  is in the class  $L_{m+p}$  (p is independent of m).

**Proof.** By the definitions

$$t(w) = g_{n+2}(g_{n+2}^{1,n}(w), g_{n+2}^{n+1}(w) + 1, f(g_{n+2}^{1,n}(w), g_{n+2}^{n+1}(w)) \cdot g_{n+2}^{n+2}(w))$$

and

$$S(\overline{x},z) = t_{\text{tier}}(s(\overline{x},0))...) = t_{\text{tier}}(\lfloor z \rfloor, g_{n+2}(\overline{x},0,1))$$

we get the property

$$\prod_{y=0}^{z} f(\overline{x}, y) = g_{n+2}^{n+2} (S(\overline{x}, z)).$$

From the defintion of the limit hierarchy we get  $\prod_{y=0}^{z} f(\overline{x}, y) \in L_{m+38}$ .

In the rest of the paper we will use the constant p as the number of limits used in the recursive defintion of the product  $\prod_{y=0}^{z} f(\overline{x}, y)$  instead of the value 38. The above constructions are tedious and can be improved with a better approximation of p.

### 4. Main results

Now we are ready to formulate two theorems which demonstrate connections between L-hierarchy and M-hierarchy.

**Theorem 4.1** Let  $f: \mathbb{R}^n \to \mathbb{R}$  be an **R**-recursive function. Then if  $f \in L_i$  then  $f \in M_{10i}$ .

**Proof.** We use a simple induction here. The case i=0 is given in Lemma 3.3. Now let us suppose that the thesis is true for i=n. Let  $f \in L_{n+1}$  be defined as  $f(\overline{x}) = \lim_{y\to\infty} g(\overline{x}, y)$  for  $g \in L_n$ . Then we can recall Theorem 4.2 from [6] which gives us the following result: to define f from g it is necessary to use at

most 10 *m*-operation. Hence for  $g \in M_{10n}$  the function **f** satisfies  $f \in M_{10n+10}$ . Similar inferences hold for  $\liminf$ ,  $\limsup$ .

Now we can give the result about the 'limit complexity' of the infimum operator *m*.

**Lemma 4.2** If  $f(\overline{x}, y): R^{n+1} \to R$  is in the class  $L_m$  then the function  $g: \mathbb{R}^n \to \mathbb{R}$ ,  $g(\overline{x}) = \mathbf{m}_y f(\overline{x}, y)$  is in the class  $L_{m+3p+9}$  is from Lemma 3.9.

**Proof.** Here we must employ the results from [6]. There we defined the function  $g: R^n \to R$ ,  $g(\overline{x}) = m_y f(\overline{x}, y)$  for  $f(\overline{x}, y): R^{n+1} \to R$  (f - **R**-recursive) replacing the *m*-operator by limit operation. First we introduced the function

This the following receives the first we introduced the function 
$$Z^f(\overline{x}, z) = \begin{cases} \inf_y \left\{ f : K^f(\overline{x}, y) = 0 \right\}, & \text{if } z = 0 \text{ and } \exists y K^f(\overline{x}, y) = 0, \\ \text{undefined} & \text{if } z = 0 \text{ and } \forall y K^f(\overline{x}, y) \neq 0, \\ 1 & \text{if } z \neq 0, \end{cases}$$

given in the following way:

defined by an integration

from 
$$S_i^f(\overline{x},t) = \int y^2 \left(1 - h^f(\overline{x},(-1)^{i+1}y - 1/2,(-1)^{i+1}y + 1/2)\right) dy$$
,  $i = 1,2$   
from  $h^f(\overline{x},a,b) = \liminf_{t \to \infty} \prod_{w=0}^{z+1} K^f(\overline{x},a+w\frac{b-a}{z})$  where  $K^f$  is the characteristic function of  $f$ .

Hence we can conclude that if  $K^f$  is in the  $L_s$  then  $Z_f$  is in the class  $L_{s+p+3}$ . Let us finish with the definition of the characteristic function of the infimum of zeros of f (see Theorem 4.2 from [5]

$$K_{\mathbf{m}}^{f}(y) = 1 - \lim_{a \to \infty} \lim_{b \to \infty} \lim_{z \to \infty} G^{f}(\overline{x}, z, a, b, y),$$

where  $G^{f}(\bar{x},z,a,b,y)$  divides the interval [a,b] into  $2^{\lfloor z \rfloor}$  equal subintervals and gives the value 1 for y from the subintervals, which contains the least zero of f in [a,b] and value 0 otherwise. Precisely for y from  $\left| a,a + \frac{b-a}{2^{\lfloor z \rfloor}} \right|$ 

$$G^{f}(\overline{x},z,a,b,y) = \begin{cases} 1, & \text{if } h^{f}(\overline{x},a,a+\frac{b-a}{2^{\lfloor z\rfloor}}) = 0, \\ 0, & \text{otherwise} \end{cases}$$

$$G^{f}\left(\overline{x},z,a,b,y\right) = \begin{cases} 1, & \text{if } h^{f}\left(\overline{x},a,a + \frac{b-a}{2^{\lfloor z \rfloor}}\right) = 0, \\ 0, & \text{otherwise} \end{cases}$$
 for  $y \in \left(a + \frac{(k-1)(b-a)}{2^{\lfloor z \rfloor}}, a + \frac{k(b-a)}{2^{\lfloor z \rfloor}}\right)$  (where  $k = 2,3,...,2^{n}$ ) we have:

$$G^{f}(\overline{x}, z, a, b, y) = \begin{cases} 1, & \text{if } \prod_{i=1}^{k-1} h^{f}\left(\overline{x}, a + \frac{(i-1)(b-a)}{2^{\lfloor z \rfloor}}, a + \frac{i(b-a)}{2^{\lfloor z \rfloor}}\right) \neq 0 \\ & \wedge h^{f}\left(\overline{x}, a + \frac{(k-1)(b-a)}{2^{\lfloor z \rfloor}}, a + \frac{k(b-a)}{2^{\lfloor z \rfloor}}\right) = 0, \\ 0, & \text{otherwise} \end{cases}$$

and for  $Y \notin [A, B]$  the function  $g_x^f$  is equal to 2.

The definition of  $G_f$  is given by the cases with respect to the value of the expression given by  $\prod h^f$ , since for  $f \in L_m$ , the function  $h_f \in L_{m+p+2}$  and  $G^f \in L_{m+2p+3}$ . Then we have  $K_m^f \in L_{m+2p+6}$ . Now we must use the function  $K_m^f$  in the same way as  $K^f$  which gives us  $Z_f$  in the class  $L_{m+3p+9}$ . The final definition of  $g(\overline{x}) = \mathbf{m}_y f(\overline{x}, y)$  ([5] Theorem 4.3) given below

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$$g(\overline{x}) = \begin{cases} Z^{f^+}(\overline{x},0) - Z^{f^-}(\overline{x},0), & \text{if } S^{f^+}(\overline{x}) < \frac{1}{12} \land S^{f^-}(\overline{x}) < \frac{1}{12}, \\ Z^{f^+}(\overline{x},0), & \text{if } \left(S^{f^+}(\overline{x}) \ge \frac{1}{12} \land S^{f^-}(\overline{x}) < \frac{1}{12}\right) \end{cases}$$
or
$$\left(S^{f^+}(\overline{x}) < \frac{1}{12} \land S^{f^-}(\overline{x}) < \frac{1}{12}\right)$$

$$\wedge Z^{f^+}(\overline{x},0) < Z^{f^-}(\overline{x},0), \qquad \text{if } \left(S^{f^+}(\overline{x}) < \frac{1}{12} \land S^{f^-}(\overline{x}) \ge \frac{1}{12}\right) \end{cases}$$
or
$$\left(S^{f^+}(\overline{x}) < \frac{1}{12} \land S^{f^-}(\overline{x}) < \frac{1}{12}\right)$$
or
$$\left(S^{f^+}(\overline{x}) < \frac{1}{12} \land S^{f^-}(\overline{x}) < \frac{1}{12}\right)$$

$$\wedge Z^{f^+}(\overline{x},0) \ge Z^{f^-}(\overline{x},0), \qquad y \ge 0, \\ 1, \qquad y < 0; \qquad f^-(\overline{x},y) = \begin{cases} f(\overline{x},-y), & y > 0, \\ 1, & y \le 0; \end{cases} \end{cases}$$
 remains the class of  $g$  identical to the class of  $Z^f$ , i.e.  $g \in L$ 

class of g identical to the class of  $Z^f$ , i.e.  $g \in L_{m+3p+9}$ .

**Theorem 4.3** Let  $f: \mathbb{R}^n \to \mathbb{R}$  be an **R**-recursive function. Then for all  $i \ge 0$  if  $f \in M_i$  then  $f \in L_{(3p+9)i}$ .

The above statement is a simple consequence of the fact  $M_0 = L_0$  and Lemma 4.2.

#### 5. Conclusions

In the paper we give the first rough approximation of 'a complexity' of limit operations in the terms of the *m*-operator and conversely. The results, interpreted in the intuitional way, can suggest what kind of connection exists between infinite limits and a *m*-operator.

We also establish the proper relation between the levels of the limit hierarchy and m-hierarchy. Let us point out that in consequence we may investigate analogies which exist for the limit hierarchy (also *m*-hierarchy) and Baire classes Pobrane z czasopisma Annales AI- Informatica http://ai.annales.umcs.pl Data: 05/11/2025 14:13:37

**R**-recursive functions is an open problem.

[7]. Also the kind of a connection between the  $\sum_{n=0}^{\infty}$  measurable functions and

## Acknowledgments

I am especially grateful to Professor Jose Felix Costa for his valuable remarks, which were used in the proof of Lemma 3.6 (definitions of p,w).

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